Problem Set 3, Math8601-Real Analysis

Assigned on Friday, Oct 1; Due on Friday, Oct 8. Autumn 2004

"Real Analysis for Real People in the Real World" – *Jackie Shen* ©

(1) (Monotonicity of l^p-norms; 10pts) In $\mathbb{R}^n$, let

$$|\vec{x}|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

denote the l^p-norm. Show that for any two positive indices p and q with $p \le q$, $|\vec{x}|_p \ge |\vec{x}|_q$ for any $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

(2) (Continuity of l^p-norms at $p = \infty$; 20pts) Following Problem (1), for $p = \infty$, define the l^∞-norm by

$$|\vec{x}|_\infty = \max(|x_1|, \cdots, |x_n|).$$

(**a**) For any $p > 0$, show that $|\vec{x}|_p \ge |\vec{x}|_\infty$. (**b**) As $p \to \infty$, show that $|\vec{x}|_p$ converges to $|\vec{x}|_\infty$ for any given $\vec{x} = (x_1, \cdots, x_n)$. (**c**) Let B_p denote the unit "ball" in $\mathbb{R}^n$ under the l^p-norm, defined by

$$B_p = \{\vec{x} \in \mathbb{R}^n \ : \ |\vec{x}|_p < 1\}.$$

Show that B_∞ is the *most spacious* unit "ball," i.e., for any positive p, $B_p \subseteq B_\infty$.

(3) (Conjugate pairs; 20pts) Two positive numbers p and q are said to be *conjugate* to each other if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(**a**) Show that $p, q \ge 1$ if they are conjugate; and that $p = 2$ is the only number that is *self-conjugate*. (**b**) For any natural number $p = n$ with $n = 1, 2, \cdots$, what is its conjugate number q? (**c**) Generally, as $p \to \infty$, where does its conjugate q go? (**d**) Let y and x be two positive numbers satisfying $y = x^{p-1}$, and q be the conjugate of p. Show that $x = y^{q-1}$. (Thus, under conjugacy, x and y are symmetric.)

(4) (Convexity; 30pts) A univariate function $g(x)$ is said to be *convex* on an interval I if for any $x, y \in I$, and any parameter $t \in [0, 1]$,

$$g(tx + (1-t)y) \le tg(x) + (1-t)g(y). \tag{1}$$

(**a**) Using the graph of $g(x)$ (i.e., x-horizontal axis vs. g-vertical axis), illustrate the meaning of the above inequality when $t = 1/2$. (**b**) Let x be an interior point of I, so that for sufficiently small $h > 0$, $x \pm h \in I$. Show that for a convex function g, the second-order central finite difference is always nonnegative:

$$\Delta_h^2 g(x) = \frac{g(x+h) + g(x-h) - 2g(x)}{h^2} \ge 0,$$

for any sufficiently small h. (**c**) Suppose g is at least *twice* differentiable on I. Apply Taylor expansion to show that $\lim_{h \to 0} \Delta_h^2 g(x) = g''(x)$ at any interior point $x \in I$. In particular, if g is convex, one must have $g''(x) \ge 0$. What does this reveal about $g'(x)$? (**d**) Suppose that $h(t)$ is a function on $t \in [0, 1]$ and at least twice differentiable, and that

$$h(0) = 0, \quad h(1) = 0, \quad \text{and} \quad h''(t) \ge 0.$$

Show that $h(t) \le 0$ for any $t \in [0, 1]$. (**e**) Show that if $g''(x) \ge 0$ for all $x \in I$, g must be convex on I. (Hint: for any fixed $x < y$, define

$$h(t) = g(tx + (1-t)y) - tg(x) - (1-t)g(y), \quad t \in [0, 1],$$

and apply the result in (**d**).)

(5) (Hölder's inequalities; 20pts) (**a**) Show that $g(s) = -\ln s$ is convex on $s \in I = (0, \infty)$. (**b**) Show that for any $t \in [0, 1]$ and $0 < X \le Y$,

$$t \ln X + (1-t) \ln Y \le \ln(tX + (1-t)Y).$$

From this, further show that for a conjugate pair p and q, one must have $xy \le \dfrac{x^p}{p} + \dfrac{y^q}{q}$, for any $0 \le x \le y$. (**c**) Based on (**b**), show that $|\langle \vec{x}\ ,\ \vec{y} \rangle| \le \frac{|\vec{x}|_p^p}{p} + \frac{|\vec{y}|_q^q}{q}$, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, and conjugate pair p and q. (**d**) Based on (**c**), show the famous inequality of Hölder:

$$|\langle \vec{x}\ ,\ \vec{y} \rangle| \le |\vec{x}|_p |\vec{y}|_q,$$

for any conjugate pair p and q, and any $\vec{x}, \vec{y} \in \mathbb{R}^n$. ($p = q = 2$ lead to the Cauchy-Schwarz inequality.)