Problem Set 4, Math8601-Real Analysis

Assigned on Friday, Oct 8; Due on Friday, Oct 15. Autumn 2004

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Convex functions and Jensen's Inequality; 20pts) Following Problem Set 3, suppose $g(x)$ is convex on an interval I, i.e., for any $x, y \in I$, and $t \in [0,1]$,

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y).$$

Writing $t_1 = t, t_2 = 1-t, x_1 = x, x_2 = y$, one has $g(t_1x_1 + t_2x_2) \leq t_1g(x_1) + t_2g(x_2)$, which looks more symmetric. Descriptively, it can be remembered as *"valuing the average $\leq$ averaging the values."*
(**a**) Show that for any *nonnegative* triples t_1, t_2, t_3 with $t_1 + t_2 + t_3 = 1$, and any $x_1, x_2, x_3 \in I$, one has

$$g(t_1x_1 + t_2x_2 + t_3x_3) \leq t_1g(x_1) + t_2g(x_2) + t_3g(x_3).$$

(**b**) By induction (i.e., going from $m-1$ to m assuming that the $m-1$-case is true), show that

$$g\left(\sum_{k=1}^{m} t_k x_k\right) \leq \sum_{k=1}^{m} t_k g(x_k), \quad \sum_{k=1}^{m} t_k = 1, \quad t_k \geq 0,$$

for any $x_k \in I$, and any finite m. (Treating t_k's as probabilities, this is *Jensen's Inequality*.)

(2) (Generalizing Hölder's Inequality; 30pts) For any two points $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ whose entries are all nonnegative: $x_i, y_i \geq 0, i = 1:n$, define

$$\langle \vec{x} \rangle = x_1 + \cdots + x_n, \quad \vec{x}^p = (x_1^p, \cdots, x_n^p), \quad \text{and} \quad \vec{x} \diamond \vec{y} = (x_1y_1, \cdots, x_ny_n) \in \mathbb{R}^n.$$

N nonnegative numbers $p_{1:m}$ are said to be *conjugate* if $p_1^{-1} + p_2^{-1} + \cdots + p_m^{-1} = 1$.
(**a**) Show that under these new notations, the inner product $\langle \vec{x}\ ,\ \vec{y} \rangle$ can be written as $\langle \vec{x} \diamond \vec{y} \rangle$. (**b**) Suppose $p_{1:m}$ are conjugate. Then for any m points $\vec{x}_{1:m}$ in $\mathbb{R}^n$, each with nonnegative entries, show that

$$\langle \vec{x}_1 \diamond \cdots \diamond \vec{x}_m \rangle \leq \frac{\langle \vec{x}_1^{p_1} \rangle}{p_1} + \cdots + \frac{\langle \vec{x}_m^{p_m} \rangle}{p_m}.$$

(**c**) Following the assumptions of (**b**), further establish the multilinear *Hölder's Inequality*:

$$\langle \vec{x}_1 \diamond \cdots \diamond \vec{x}_m \rangle \leq |\vec{x}_1|_{p_1} \cdots |\vec{x}_m|_{p_m}.$$

(3) (Closed sets vs. compact sets; 20pts) Let (K_n) be a shrinking chain of *nonempty closed* subsets of $\mathbb{R}^n$:

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$$

(**a**) Construct an example showing the possibility of $\bigcap_{n=1}^{\infty} K_n = \phi$, the empty set.
(**b**) Show that $\bigcap_{n=1}^{\infty} K_n$ is *never* empty if (K_n) is a shrinking chain of nonempty *compact* subsets.
(**c**) Following (**a**), suppose in addition that there exists some n, such that K_n has finite diameter. By *Heine-Borel's Compactness Theorem*, show that the infinite intersection then must be non-empty.

(4) (Topology vs. p-norms in $\mathbb{R}^n$; 10pts) In the lecture, it is defined that, a subset $Q \subseteq \mathbb{R}^n$ is said to be *open* if for any point $x = (x_1, \cdots, x_n) \in Q$, there exists some $r > 0$, such that the ball $B_r(x) \subseteq Q$. Notice that here the "ball" refers to the Euclidean ball define by

$$B_r(x) = \{y = (y_1, \cdots, y_n) \in \mathbb{R}^n \ :\ |y - x| = \sqrt{(y_1 - x_1)^2 + \cdots (y_n - x_n)^2} < r\}.$$

Our classmate Trinity however recommends to replace the Euclidean norm (i.e., the 2-norm) by a general p-norm for some fixed $p \in [1, \infty]$, and use the p-ball $B_r^p(x) = \{y \ :\ |y - x|_p < r\}$ instead in the definition. Show that in fact the definition of *openness* is *independent* of the particular p-norm one uses. That is, a set Q is *open* by the 2-ball definition if and only if it is *open* by the p-ball definition.

(5) (An application of compactness; 20pts) By *Heine-Borel's Compactness Theorem*, the closed unit interval $I = [0,1]$ is compact in $\mathbb{R}^1$. Let (I_n) be a σ-covering of I with *open* intervals $I_n = (a_n, b_n)$. Show that

$$1 \leq \sum_{n=1}^{\infty} (b_n - a_n).$$

That is, length$(I) \leq \sum_n$ length(I_n), which looks so intuitively sound, but you have to rigorously prove it. [Hints: Heine-Borel Theorem reduces the problem to the finite-covering situation, which can then be established by an induction argument: 1-interval covering, 2-interval covering, ...]