Problem Set 5, Math8601-Real Analysis

Assigned on Friday, Oct 15; Due on Friday, Oct 22. Autumn 2004

"Real Analysis for Real People in the Real World" – *Jackie Shen* ©

(1) (Application of Heine-Borel's Compactness Theorem; 10pts) In $\mathbb{R}^d$, suppose a chain of non-empty *open* sets $Q_1 \supseteq Q_2 \supseteq \cdots$ have the *boundary repelling property* (BRP): for any n, there exists some $m > n$, such that $\overline{Q_m} \subseteq Q_n$, where $\overline{Q_m}$ denotes the closure of Q_m. (**a**) Suppose in addition Q_1 is bounded. Show that $\cap_{n=1}^{\infty} Q_n \neq \phi$, nonempty. (**b**) Construct an example in $\mathbb{R}^2$ for which Q_1 is unbounded but the infinite intersection is empty, even though the BRP is satisfied.

(2) (Equivalence of Topological Compactness and (Sequential) Compactness; 20pts) Let us polish the proof in the lecture of the following direction: if $K \subseteq \mathbb{R}^d$ is T.C., K must be (sequentially) compact as well. Suppose otherwise there exists a sequence $(x_n)_{n=1:\infty}$ which has *no* convergent subsequence. Define $A = \{x_n \mid n = 1 : \infty\}$. Show that
(2.**a**) A must be an infinite set;
(2.**b**) A must be *closed*, i.e., for any $x \notin A$, there exists some $r > 0$, such that $B_r(x) \cap A = \phi$;
(2.**c**) A must be an *isolated* set: for any $a \in A$, there exists some $r_a > 0$, such that $B_{r_a}(a) \cap A = \{a\}$ only;
(2.**d**) Since any closed set of a T.C. set is still T.C. (see the lecture), (**b**) implies that A is T.C. Show that (**a**) and (**c**) however imply that there exists an infinite open cover of A for which no finite subcover (of A) is possible. (Thus K must be (sequentially) compact.)

(3) (Elementary Approaches to Boxes (i.e., do not use any measure theory; required); 30pts) Let $A = \{a_1 < a_2 < \cdots < a_n\}$ and $C = \{c_1 < c_2 < \cdots < c_m\}$ denote any two finite collections of ordered real numbers. The *net* induced by the ordered pair (A, C), denoted for convenience by $A \otimes C$, refers to the following collection of closed boxes (or rectangles) in $\mathbb{R}^2$:

$$A \otimes C = \{I = [a_i, a_{i+1}] \times [c_j, c_{j+1}] \ : \ i = 1 : n-1, j = 1 : m-1\}.$$

(3.**a**) Let $a = a_1, b = a_n, c = c_1, d = c_m$, and $Q = [a, b] \times [c, d]$. Show that $|Q| = \sum_{I \in A \otimes C} |I|$.
(3.**b**) Suppose $I_1, \cdots, I_M$ are non-overlapping closed boxes in a larger box $Q = [a, b] \times [c, d]$. Show that $\sum_{i=1:M} |I_i| \leq |Q|$. [Hints. Suppose $I_i = [a_1^i, a_2^i] \times [c_1^i, c_2^i]$. Define $A = \{a_j^i \mid i = 1 : M, j = 1 : 2\}$ and $C = \{c_j^i \mid i = 1 : M, j = 1 : 2\}$. Then apply (**a**).]
(3.**c**) Using the same *net generation* technique, show that for any closed boxes $I_{1:M}$ and Q, if $Q \subseteq I_1 \cup \cdots \cup I_M$, one must have $|Q| \leq \sum_{i=1:M} |I_i|$.
(3.**d**) Let $(I_n)_{n=1:\infty}$ denote a countable collection of *open* boxes that cover $Q = [a, b] \times [c, d]$. Combining *Heine-Borel's Compactness Theorem* and (**c**), show that $|Q| \leq \sum_{n=1:\infty} |I_n|$. (This generalizes Problem 5 in Problem Set 4 to high dimensions.)

(4) (Multiscale (Dyadic) Zooming Technique; 20pts) Based on the pictorial proof in Lecture 10, write up the complete proof for the following theorem in Lecture 10, as clearly but also concisely as possible:
Theorem. *Any* open *set* $G \subseteq \mathbb{R}^d$ *can be expressed as a countable union of non-overlapping closed boxes.*

(5) (Atomic Sets and Their Induced Outer Measures; 10pts) Given a space X, consider the two outer measures μ_1 and μ_2 induced separately by two atomic collections (Ω_1, ρ_1) and (Ω_2, ρ_2), where both $\Omega_{1:2}$ σ-cover X and contain the empty set ϕ, and $\rho_{1:2}$ are nonnegative and $\rho_{1:2}(\phi) = 0$. (See our Lecture 5.) Suppose for any $Q \in \Omega_1$, there exists a countable (including finite) sequence $(Q_n)_{n=1:\infty}$ in Ω_2, such that $Q = \cup_{n=1:\infty} Q_n$, and

$$\rho_1(Q) = \sum_{n=1:\infty} \rho_2(Q_n). \tag{1}$$

Show that for any set $A \subseteq X$, $\mu_2(A) \leq \mu_1(A)$. [This result is crucial for the next problem.]

(Problem 6 is on the other side.)

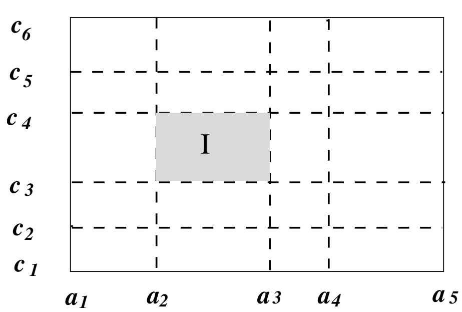
(6) (Independence (of the Lebesgue Outer Measure) on Atomic Sets; 30pts) In $\mathbb{R}^2$, define

$$\begin{aligned}
\Omega &= \{I = [a,b]\times[c,d] \ : \ a \le b, c \le d\} = \text{all closed boxes;}\\
\Omega_h &= \{I = [a,b]\times[c,d] \ : \ a \le b, c \le d; b-a \le h, d-c \le h\} = \text{all closed boxes whose sides} \le h;\\
\Omega_c &= \{C = [a,a+L]\times[c,c+L] \ : \ a,c \in \mathbb{R}, L \ge 0\} = \text{all closed cubes or squares;}\\
\Omega_r &= \{R = [a,a+4L]\times[c,c+4L] \setminus [a+L,a+3L]\times[c+L,c+3L] \ : \ a,c\in\mathbb{R}, L\ge 0\} \cup \{\{x\} \ : \ x \in \mathbb{R}^2\}.
\end{aligned}$$

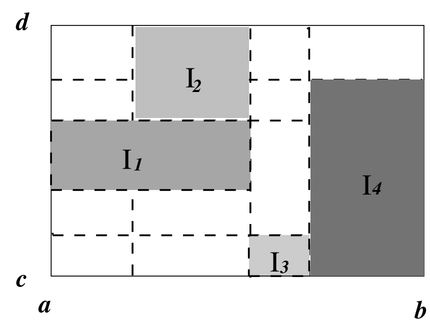
Here the last collection consists of all singletons as well as square rings. Define the ordinary volume

$$|I| = (b-a)\times(d-c), \quad |C| = L^2, \quad |R| = (4L)^2 - (2L)^2 = 12L^2,$$

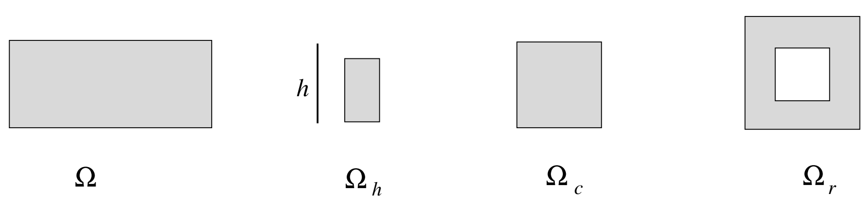
for each typical atom in the above atomic collections. Then in the lecture (as well as in the book by Wheeden-Zygmund), we have employed the first atomic collection $(\Omega, |I|)$ to define the Lebesgue outer measure. Applying the result of the preceding problem, show *separately* that the three outer measures, μ_h, μ_c, μ_r, generated by the other three atomic collections $(\Omega_h, |I|)$, $(\Omega_c, |C|)$, and $(\Omega_r, |R|)$, are all equal to the Lebesgue outer measure.



Problem 3(a)



Problem 3(b)



Problem 6