# Problem Set 6, Math8601-Real Analysis

Assigned on Friday, November 5; Due on Friday, November 12.    Autumn 2004

*"Real Analysis for Real People in the Real World"* – *Jackie Shen* ©

(1) (Vitali's non-measurable set; 30pts) Let us complete the construction of non-measurable sets (in $\mathbb{R}^1$) in Lecture 12. A set $A \subseteq \mathbb{R}^1$ is said to be *positive* if it has positive Lebesgue outer measure: $|A|_e > 0$. Let $\mathbb{Q} = \{q_n \mid n = 1 : \infty\}$ denote all rationals in $\mathbb{R}^1$. Recall that a set $E \subseteq \mathbb{R}^1$ is called a Vitali set if (i) for any $x \in \mathbb{R}^1$, there exists some $y \in E$, $x - y \in \mathbb{Q}$, and (ii) for any two distinct $x, y \in E$, $x - y \notin \mathbb{Q}$. (Zermelo's Axiom of Choice guarantees the existence of such a set $E$.) For any $a \in \mathbb{R}^1$, define $A_a = A + a$.

- (1.a) Show that $\mathbb{R}^1 = \cup_{q \in \mathbb{Q}} E_q$ is a disjoint partitioning.
- (1.b) Show that $E$ must be positive (using the translation invariance of the Lebesgue outer measure).
- (1.c) Define $E - E = \{a - b \mid a, b \in E\}$. Show that $(E - E) \cap \mathbb{Q} = \{0\}$.
- (1.d) To show that $E$ cannot be measurable, we need the following key lemma: For any positive measurable set $A$, $A - A$ must cover an open interval that contains 0, to be proven in the next problem.

(2) (Positive measurable sets; 20pts) Suppose $E$ is a positive set in $\mathbb{R}^1$ (no assumption on measurability).

- (2.a) Show that $\sup_I \dfrac{|I \cap E|_e}{|I|} = 1$, where $I$ runs over all non-empty open intervals (hint: using exterior open-set approximation to $E$).
- (2.b) Let $I = (0, 1)$ and $A \subseteq I$ be measurable and $|A|/|I| > 1/2 + \delta/2$ for some positive $\delta$. Show that for any $t : |t| < \delta$, $A \cap A_t \neq \phi$ (hint: using shift-invariance and $|I \cup I_t| \leq 1 + |t|$).
- (2.c) Following (2.b), show that the open interval $(-\delta, \delta) \subseteq A - A$.
- (2.d) Based on (a, b, c), show that if $E$ is positive and measurable, then for some $\delta > 0$, $(-\delta, \delta) \subseteq E - E$. In particular, $E - E$ contains "lots of" rationals, and the Vitali set cannot be measurable (by (1.c)).

(3) (Let's play with Vitali, 20pts) Let $E$ be a Vitali set.

- (3.a) Based on $E$, for any $\epsilon > 0$, construct a non-measurable set $F$ such that $|F|_e < \epsilon$.
- (3.b) Show that any positive subset of $E$ is still non-measurable.
- (3.c) Based on $E$, construct a non-measurable set $F$ which has a positive and measurable subset.
- (3.d) Show that any positive measurable set $G$ must contain a non-measurable subset. (Hint: Define $G_q = G \cap E_q$ for $q \in \mathbb{Q}$, and apply (3.b).)

(4) (Measurable functions, 10pts) Suppose $f, g : X \to \mathbb{R}^1$ are both $\Sigma$-measurable. Show that $h(x) = f(x)g(x)$ is also $\Sigma$-measurable. Here $\Sigma$ is a given $\sigma$-algebra on $X$.

(5) (Measurable v.s. non-measurable, 20pts) Let $(X, \Sigma) = (\mathbb{R}^1, \Sigma_L)$ be the Lebesgue algebra in $\mathbb{R}^1$.

- (5.a) Construct an example to show that, given $f$ and $g$ both non-measurable, it is still possible for $f + g$ and $f(x)g(x)$ both(!!) being measurable.
- (5.b) Let $\phi(x) = ax + b$ for some $a \neq 0$ and $b \in \mathbb{R}^1$. Suppose $f$ is Lebesgue non-measurable. Show that $h(x) = \phi(f(x)) = af(x) + b$ is also Lebesgue non-measurable.
- (5.c) Now let $\phi(x) = ax^2 + bx + c$ be a parabola with $a \neq 0$. Construct an example (including choosing $a, b, c$) to show that, given $f$ non-measurable, it is still possible for $h(x) = \phi(f(x))$ being measurable.