

Problem Set 7, Math8601-Real Analysis

Assigned on Friday, November 12; Due on Friday, November 19. Autumn 2004

“Real Analysis for Real People in the Real World” – Jackie Shen ©

(1) (Dropping finiteness from Egorov to Lusin; 20pts) Let (F_n) be a sequence of *closed* sets in \mathbb{R}^2 .

- (a) Construct an example showing that $F = \cup_{n=1}^{\infty} F_n$ could be non-closed.
- (b) Suppose that the sequence has the following *escaping* property:

$$\text{dist}(\mathbf{0}, F_n) \rightarrow \infty, \quad n \rightarrow \infty; \quad \mathbf{0} = (0, 0).$$

Then show that $F = \cup_{n=1}^{\infty} F_n$ must be still closed.

- (c) Recall that a function $f : E \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^1$ defined on a measurable set E is continuous if and only if for any closed set $F \subseteq \mathbb{R}^1$, $f^{-1}(F)$ is relatively closed in E (i.e., there exists a closed set $A \subseteq \mathbb{R}^2$ such that $f^{-1}(F) = A \cap E$.) Suppose now that (F_n) has the escaping property, and that each restriction $f|_{F_n}$ is continuous. Show that $f|_{\cup_n F_n}$ must be continuous as well. (This explains why when going from Egorov to Lusin, the finiteness condition on E can be dropped after properly partitioning E .)

(2) (How far one could go from Lusin; 20pts) Suppose f is measurable on a measurable Lebesgue set $E \in \mathbb{R}^d$.

- (a) Using Lusin’s Theorem, show that there exists a chain of *increasing* closed subsets of E : $F_1 \subseteq F_2 \subseteq \dots$, such that $|E - F_n| \rightarrow 0$ as $n \rightarrow \infty$, and each $f|_{F_n}$ is continuous.
- (b) Construct an example in \mathbb{R}^1 , for which (F_n) is an increasing chain of closed sets, and each $f|_{F_n}$ is continuous, but $f|_{\cup_n F_n}$ is not continuous. (Hint: consider the Heaviside function $H(x)$.)
- (c) The key reason for (b) to occur is that the local topology is constantly changed as new F_n ’s join in. To remedy the problem, we say a sequence (unnecessarily being increasing) of closed sets (F_n) is *locally consistent* (L.C.) if for any finite ball $B_r(x)$ in \mathbb{R}^d , there exists some N , such that

$$B_r(x) \cap (\cup_{n=1}^{\infty} F_n) = B_r(x) \cap (\cup_{n=1}^N F_n).$$

Thus intuitively, locally to x , $F_{n \geq N+1}$ ’s add no new information at all. Show that $f|_{\cup_n F_n}$ must be continuous if (F_n) is an L.C. sequence of closed sets.

- (d) Show that any *escaping* sequence (F_n) of closed sets must be L.C. In addition, construct an L.C. sequence that is non-escaping.

(3) (Finiteness condition in Egorov; 10pts) Let $H(x)$ be the Heaviside 0-1 function and $f_n(x) = H(x-n)$, $n = 1 : \infty$ defined on $E = \mathbb{R}^1$. Since E does not have finite Lebesgue measure, the result of Egorov’s ϵ -Uniform Convergence Theorem understandably fails to hold. We now define instead the *Cauchy measure* on \mathbb{R}^1 by:

$$\mu_c(a, b) = \int_a^b \frac{dx}{1+x^2} = \arctan x \Big|_a^b, \quad \forall \quad a < b.$$

Later we shall learn that indeed μ_c is a measure for all Lebesgue measurable sets. Calculus shows that $\mu_c(\mathbb{R}^1) = \pi$, which is finite! Therefore, one should expect that Egorov’s Theorem holds with respect to μ_c . That is, for any $\epsilon > 0$, there exists a closed set F , such that $\mu_c(E - F) < \epsilon$, and f_n ’s converge uniformly on F to $f \equiv 0$. Question: for the above Heaviside sequence (f_n) , for any given $\epsilon > 0$, explicitly construct such a closed set F .

(4) (Uniqueness of limits; 10pts) Let E be a measurable set in \mathbb{R}^d , and (f_n) a sequence of measurable functions on E . Suppose that on E , $f_n \xrightarrow{a.e.} f$, and $f_n \xrightarrow{m.} g$. Show that $f = g$, a.e. on E .

(5) (Convergence in measure (c.i.m) v.s. convergence a.e.; 20pts) Let $E_n \subseteq E = [0, 1]$ be a sequence of measurable sets such that (i) $|E_n| \rightarrow 0$ as $n \rightarrow \infty$, (ii) $\overline{\lim}_{n \rightarrow \infty} E_n = [0, 1]$. For each n , define $f_n = 1_{E_n}(x)$ to be the characteristic function of E_n in $E = [0, 1]$. Show that (a) f_n ’s converge to $f \equiv 0$ on E in measure, but (b) a.e. on E , f_n ’s do **not** converge to $f = 0$. Furthermore, (c) construct a sequence of intervals I_n ’s in E that satisfy both conditions (i) and (ii).

(6) (Enhancing the proof from c.i.m. to c.a.e.; 20pts) (f_n) ’s are measurable on a measurable set $E \subseteq \mathbb{R}^d$.

- (a) In any measure space (X, Σ, μ) , prove that for any N , “ $\sum_{m \geq N} \mu(E_m) < \infty \Rightarrow \mu(\overline{\lim}_m E_m) = 0$.”
- (b) Show that “ $f_m \xrightarrow{a.e.} f$ on $E \Leftrightarrow$ for any $\epsilon > 0$, the “bad” set $B_\epsilon = \overline{\lim}_m \{|f_m - f| \geq \epsilon\}$ is a null set.”
- (c) For any $m, n \geq 1$, define the “bad” set $B_m(n) = \{|f_n - f| \geq m^{-1}\}$. Suppose $f_n \xrightarrow{m.} f$. Then for any fixed m , there exists some n_m , such that $|B_m(n_m)| \leq m^{-2}$. Make proper choices so that $n_1 < n_2 < \dots$. Question: based on (a) and (b), show that $f_{n_m} \xrightarrow{a.e.} f$ as $m \rightarrow \infty$.