Problem Set 8, Math8601-Real Analysis

Assigned on Monday, November 22; Due on Friday, December 3.   Autumn 2004

"Real Analysis for Real People in the Real World" – Jackie Shen ©

In this problem set, we shall always assume, (i) any given function f is nonnegative and measurable, and could take ∞, i.e., $f \in [0, \infty]$; (ii) any given set is measurable; (iii) the underlying measure space $(\mathbb{R}^d, \Sigma, \mu)$ is Radon in the sense that for any bounded ball B_R with radius R, $\mu(B_R) < \infty$. The Lebesgue measure $d\mu = dx$ of course satisfies the condition; and (iv) $\mathcal{P}(E)$ denotes all finite partitions of a measurable set E, and a partition $\pi \in \mathcal{P}(E)$ is a finite number of disjoint (measurable) subsets $\pi = \{E_1, \cdots, E_N\}$ whose union is E. Recall that in our lecture the integral is defined as

$$\int_E f d\mu = \sup_{\pi \in \mathcal{P}(E)} \langle f \ , \ \pi \rangle, \quad \text{with } \langle f \ , \ \pi \rangle = \langle f \ , \ \pi \rangle_{\text{inf}} = \sum_{A \in \pi} \left(\inf_A f \right) \mu(A).$$

(1) (Play with partitions; 15pts)

- (a) For $\pi, \delta \in \mathcal{P}(E)$, let $\pi \wedge \delta \in \mathcal{P}(E)$ be the *intersection* partition. Show that $\langle f \ , \ \pi \wedge \delta \rangle \geq \langle f \ , \ \pi \rangle$.
- (b) For any $A \subseteq E$, let $\pi \wedge A$ denote the *induced* partition of A. For any $\alpha \in \mathcal{P}(A)$ and $\beta \in \mathcal{P}(B)$ with disjoint A and B, let $\alpha \vee \beta$ denote the *union* partition of $A \cup B$. Show the following identity: for any $\pi \in \mathcal{P}(E)$ and $A \subseteq E$,

$$(\pi \wedge A) \vee (\pi \wedge (E \setminus A)) = \pi \wedge \{A, E \setminus A\}, \quad \text{where } \{A, E \setminus A\} \text{ is a binary partition.}$$

- (c) For any $\pi = \{E_1, \cdots, E_N\} \in \mathcal{P}(E)$ and $A \subseteq E$, define the *inherited* infimum sum by

$$\langle f \ , \ \pi \mid A \rangle = \sum_{n=1}^{N} \left(\inf_{E_n} f \right) \mu(E_n \cap A).$$

  Show that

$$\langle f \ , \ \pi \mid A \rangle \leq \langle f \ , \ \pi \wedge A \rangle; \quad \text{and } \langle f \ , \ \pi \rangle = \langle f \ , \ \pi \mid A \rangle + \langle f \ , \ \pi \mid E \setminus A \rangle.$$

(2) (Absolute continuity (A.C.); 20pts) The remarkable A.C. property says: if $\int_E f d\mu < \infty$, then for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any $A \subseteq E$ with $\mu(A) < \delta$, one has $\int_A f d\mu < \varepsilon$. Let us solidify our proof in the lecture.

- (a) Show that if $\int_E f d\mu - \langle f \ , \ \pi \rangle < \varepsilon$, then $\int_A f d\mu - \langle f \ , \ \pi \mid A \rangle < \varepsilon$ for any $A \subseteq E$.
- (b) Define $M = M_\pi = \max_{B \in \pi : \mu(B) > 0} \left(\inf_B f \right)$. Show that $M < \infty$ for any $\pi \in \mathcal{P}(E)$.
- (c) Establish the A.C. property by noticing that $\langle f \ , \ \pi \mid A \rangle \leq M_\pi \cdot \mu(A)$.
- (d) Let $E = (0, 1) \subseteq \mathbb{R}^1$, and $d\mu = dx$ be the Lebesgue measure. Consider the particular function $f(x) = 1/\sqrt{x}$ on E. First show that $\int_E f dx < \infty$, and then, for the particular value $\varepsilon = 0.01$, find a particular positive value δ in the A.C. property and justify your answer.

(3) (Play with the ceiling operator; 30pts) For any $M > 0$, the ceiling operator $[\cdot]_M$ is defined for any $a \geq 0$ by $[a]_M = \min(a, M)$. For any nonnegative measurable function $f(x)$ on E, we define its *ceiled* version $[f]_M$ by $[f]_M(x) = [f(x)]_M, \forall\, x \in E$. The ceiling operator is very useful in the integration theory.

- (a) (Monotonicity; 5pts) Show that $g \leq f$ implies $[g]_M \leq [f]_M$.
- (b) (10pts) Show that $f_n \xrightarrow{a.e.} f$ on E if and only if for any $M > 0$, $[f_n]_M \xrightarrow{a.e.} [f]_M$.
- (c) (Shrinkage; 5pts) Show that for any $a, b \geq 0$, $|[b]_M - [a]_M| \leq |b - a|$.
- (d) (10pts) Suppose that $f, f_n < \infty, a.e.$ on E, $n = 1 : \infty$. Show that $f_n \xrightarrow{m.} f$ on E implies that for any $M > 0$, $[f_n]_M \xrightarrow{m.} [f]_M$. Construct an example showing that the opposite could be wrong (hint: when you ceil a room, you cannot see what's going on above the roof).

(4) (Finite approximation to integrals and M.C.T.; 10pts) In the lecture, we established the finite approximation theorem:

$$\int_{E_R} [f]_M d\mu \to \int_E f d\mu, \quad R, M \to \infty, \quad E_R = E \cap B_R,$$

where B_R is the ball in $\mathbb{R}^d$ of radius R and centered at the origin. Based on this approximation, (3.c), and Egorov's ε-Uniform Convergence Theorem, establish the *Monotone Convergence Theorem* (M.C.T.): if $0 \leq g_n \uparrow f$ on E (i.e., monotonically increasing to f), then $\int_E g_n d\mu \to \int_E f d\mu$, $n \to \infty$.

(5) (Egorov, absolute continuity, and L.D.C.T.; 10pts) Suppose $\mu(E) < \infty$, $f_n \xrightarrow{a.e.} f$, and $0 \leq f_n \leq F$ on E for some measurable function $F(x)$ with finite integral $\int_E F d\mu < \infty$. Establish *Lebesgue's Dominated Convergence Theorem* (L.D.C.T.) for $\mu(E) < \infty$: $\int_E f_n d\mu \to \int_E f d\mu$, by combining (required !!) the absolute continuity property of F and Egorov's ε-Uniform Convergence theorem.

(6) (From F.L. to L.D.C.T.; 10pts) Apply (required !!) Fatou's Lemma to prove the general L.D.C.T. (i.e., without assuming $\mu(E) < \infty$ as in the preceding problem).

(7) (Mechanisms for the strict inequality in F.L.; 15pts) Fatou's Lemma says, for any nonnegative sequence of measurable functions f_n on E,

$$\int_E \underline{\lim}_n f_n d\mu \leq \underline{\lim}_n \int_E f_n d\mu.$$

In particular, suppose $f_n \xrightarrow{a.e.} f$ on E. Then one has

$$\int_E f d\mu \leq \underline{\lim}_n \int_E f_n d\mu.$$

Let us see how the *strict* inequality could be realized.

(a) (**horizontal escaping of mass**) Let $E = \mathbb{R}^1$ and $d\mu = dx$ be Lebesgue. For $n = 1 : \infty$, let $f_n = H(x-n) - H(x-n-1)$ be a moving box of length 1 and height 1, where $H(x) = 1_{[0,\infty)}(x)$ denotes the Heaviside function. Show that the strict inequality in F.L. holds for this sequence.

(b) (**vertical escaping of mass**) Let $E = [0, 1]$ and $d\mu = dx$ be Lebesgue. Let $g_n = n \times 1_{[0,\frac{1}{n}]}(x)$ be the sequence of growing (towards the sky) boxes with fixed areas 1. Show that the strict inequality holds for this sequence as well.

(c) (**bounding and ceiling**) Inspired by these two escaping (or loss) mechanisms, suppose we attempt to prevent them by imposing: (i) (bounding) $\mu(E) < \infty$; and (ii) (ceiling) there exists some $M > 0$ such that $h_n \in [0, M]$, $n = 1 : \infty$. Suppose $h_n \xrightarrow{a.e.} h$. Show that then the *equality* must hold in F.L. for this sequence.

Happy Holidays from Jackie!