Problem Set 9, Math8601-Real Analysis

Last problem set of Autumn 2004

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Of review nature; 10pts) By noticing that $\overline{\lim}(-a_n) = -\underline{\lim} a_n$, state the limsup-version of *Fatou's Lemma*, including its proper conditions, for a general sequence of signed measurable functions (f_n) on E.

(2) (Of review nature; 10pts) For a general sequence of signed measurable functions (f_n) which *decrease* monotonically on E, state the *Monotone Convergence Theorem* and its proper conditions.

(3) (Of review nature; 10pts) Let $E = \mathbb{R}$ and $d\mu = dx$ be the Lebesgue measure. Suppose $g_n \equiv -1/n, n = 1 : \infty$ are constant functions on E and $g \equiv 0$. Then $g_n \uparrow g$ on E. Explain the lack of which condition makes this sequence violate the *Monotone Convergence Theorem.*

(4) (A classical application of Lebesgue's Dominated Convergence; 10pts) Suppose f is differentiable *everywhere* on $(-0.1,\ 1.1)$, and for some constant $L > 0$, $|f'(x)| \le L$ for all $x \in (-0.1, 1.1)$. Define

$$I(h) = \int_0^1 f(x+h)dx, \qquad |h| < 0.1.$$

Show that $I'(0) = \int_0^1 f'(x)dx$. That is, symbolically, $\left.\frac{d}{dh}\right|_{h=0}$ commutes with the integration operator. [Notice that we are not assuming f' is continuous. Also notice that from Calculus, as long as f is continuous, one has $I'(0) = f(1) - f(0)$. Do you still remember how to prove it? (not required)]

(5) (2-D version Problem 4 -copied from Wheeden-Zygmund; 10pts) Let $Q = (0,1)^2$ be the open square. Suppose $f(x,y)$ is Lebesgue integrable on $(0,1)$ with respect to y for any given $x \in (0,1)$. In addition, assume that the partial derivative $f_x(x,y) = \partial f/\partial x(x,y)$ exists everywhere on Q, and is bounded: $|f_x(x,y)| \le L$ for some $L > 0$. Apply *Lebesgue's Dominated Convergence Theorem* to show that for any $x \in (0,1)$,

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy.$$

[**In the next semester, we shall learn more about differentiations. These two give you a taste.**]

(6) (Riemann integrability is equivalent to a.e. continuity: part I; 20pts) Let $R[a,b]$ denote the collection of all Riemann integrable and bounded functions. For any given finite partition of $I = [a,b]$,

$$\Gamma = \{a_0 = a < a_1 < \cdots < a_{n-1} < a_n = b\}, \quad \text{with } I_k = (a_k, a_{k+1}),$$

in the lecture we have defined the lower and upper functions f_Γ^- and f_Γ^+ to be the step functions with

$$f_\Gamma^-(x) = \inf_{I_k} f, \quad \text{and } f_\Gamma^+(x) = \sup_{I_k} f, \quad x \in I_k, \quad k = 0 : n-1.$$

Then by definition, a bounded function f belongs to $R[a,b]$ if and only if there exists a finite number A such that $\int_I f_\Gamma^\pm \to A$ as $|\Gamma| \to 0$. In this problem we show that $f \in R[a,b]$ implies f is a.e. continuous.

(a) First show the following equivalence result in Calculus. A function $g(x)$ is continuous at x_0 if and only if for any $\varepsilon > 0$, there exists some *open* interval I_ε *containing* x_0, such that

$$g_{I_\varepsilon}^+ - g_{I_\varepsilon}^- < \varepsilon, \quad \text{with } g_{I_\varepsilon}^+ = \sup_{I_\varepsilon} g(x) \quad \text{and } g_{I_\varepsilon}^- = \inf_{I_\varepsilon} g(x).$$

That is, show that this statement is equivalent to the classical ε-δ description of continuity.

(b) As in the lecture, let (Γ^n) be any sequence of refined partitions with (i) $|\Gamma^n| \to 0$ and (ii) $\Gamma^n \subseteq \Gamma^{n+1}$. Let $f_n^\pm$ denote the upper and lower functions associated with Γ^n. Due to the (a.e.) monotonicity, suppose $f_n^- \uparrow f^-$ and $f_n^+ \downarrow f^+$. Then $f^- \le f \le f^+$ (a.e.), and $f \in R[a,b]$ implies that

$$f^-(x) = f(x) = f^+(x), \quad a.e.$$

Define three "bad" sets by $B_0 = \cup_{n=1}^\infty \Gamma^n$, $B_+ = \{x \in [a,b] : f_n^+ \nrightarrow f\}$, $B_- = \{x \in [a,b] : f_n^- \nrightarrow f\}$. Show that $B = B_0 \cup B_+ \cup B_-$ is a Lebesgue null set.

(c) Suppose $f \in R[a,b]$. Show that f must be continuous at any $x \in [a,b] \setminus B$ (by using (a)).

(7) (Riemann integrability is equivalent to a.e. continuity: part II; 20pts) We now show the opposite: if f is bounded on $[a, b]$, and a.e. continuous, then $f \in R[a, b]$. It suffices to only show that there exists a finite value A, such that for any sequence of partitions (Γ^n) with $|\Gamma^n| \to 0$, both $\int_a^b f_n^+, \int_a^b f_n^- \to A$.

(a) This time we single out two potential "troublemakers:"

$$B_0 = \cup_{n=1}^{\infty} \Gamma^n, \quad \text{and } B_* = \{x \in [a, b] : f \text{ is not continuous at } x\},$$

and define the bad set $B = B_0 \cup B_*$. B is clearly a null set (assuming that f is a.e. continuous). Show that both $f_n^{\pm}$ converge to f for any $x \in [a, b] \setminus B$.

(b) Since f is assumed to be bounded, show that there exists some constant $F > 0$, such that $|f_n^{\pm}(x)| \leq F$ for any n and $x \in [a, b] \setminus B$ (since we do not define nor care about the values on B_0 for $f_n^{\pm}$'s). Then apply *Lebesgue's Dominated Convergence Theorem* to finish the proof. What is the number A?

(c) In addition, can you apply only the original definition of *measurability* to show directly that an a.e. continuous and bounded function f on $[a, b]$ must be Lebesgue measurable. (The proof should be at most several lines long.)

(8) (Foundation of Modern Probability Theory: Cumulative level sets F_λ and the distribution function $\omega_f(\lambda)$; 10pts) Suppose a measure space (X, Σ, μ) is a *probabilistic* space in the sense that the total measure (or *certainty*) $\mu(X) = 1$ (or 100% *total certainty*). Given a measurable function f on X, recall that in the lecture, we have defined the *cumulative* level sets F_λ and the (cumulative) distribution function $\omega_f(\lambda)$ by $\boxed{F_\lambda = \{x \in X \mid f(x) < \lambda\}, \quad \text{and } \omega_f(\lambda) = \mu(F_\lambda), \quad \lambda \in (-\infty, \infty).}$ We have proven the remarkable equivalence formula: assuming that $A \leq f < B$, then

$$\int_X f d\mu = \int_A^B \lambda \, d\omega_f(\lambda),$$

which is the foundation of the entire probability theory! When $d\mu = dx$, this is often called the equivalence between Lebesgue and Riemann-Stieltjes since the right hand side is an R.-S. integral. Let us play with this formula on a concrete example.

Let $X = B^2$ denote the unit *open* disk in $\mathbb{R}^2$ centered at the origin, and $d\mu = \frac{1}{\pi} dx = \frac{1}{\pi} dx_1 dx_2$ the normalized Lebesgue measure so that $\mu(X) = 1$. Consider the function $f(x) = |x| = \sqrt{x_1^2 + x_2^2}$, for which one can take $A = 0$ and $B = 1$.

(a) Apply the polar coordinates to calculate $L = \int_X f(x) d\mu$, as in Calculus.

(b) Show that the (cumulative) distribution function of f is $\omega_f(\lambda) = \lambda^2$ for $\lambda \in [A, B]$, quite nice. Calculate $R = \int_A^B \lambda \, d\omega_f(\lambda)$. Is L equal to R?

Happy Holidays from Jackie!
My Honor and Pleasure Teaching You All Great Students.