Problem Set 10, Math8602-Real Analysis

Assigned on Friday, Jan 28, 2005; Due on Friday, Feb 4, 2005

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Fubini vs. Tonelli; 10pts) What is difference between the assumptions of Fubini's Theorem and Tonelli's Theorem? Show that they are equivalent.

(2) (Tonelli/Fubini fails for "weird" measures (from Lieb-Loss); 10pts) Let $I = J = [0, 1]$. Endow I with the ordinary Lebesgue measure dx on the Lebesgue σ-algebra, while J with the counting measure μ defined for any subset. Let $f(x, y) = 1_{x=y}(x, y)$ be the indicator of the diagonal line in $I \times J$. Show that

$$\int_J d\mu(y) \int_I f(x, y)dx = 0 \neq 1 = \int_I dx \int_J f(x, y)d\mu(y).$$

Thus Tonelli's Theorem fails - the "counting" measure is too "weired." Show that the counting measure is not σ-finite.

(3) (Tensor products of sets (modified from Wheeden-Zygmund); 20pts) Let $E_1 \subseteq \mathbb{R}^n$ and $E_2 \subseteq \mathbb{R}^m$ be two sets.

(a) By definition show that $|E_1 \times E_2|_e \le |E_1|_e \times |E_2|_e$ (in terms of outer measures).

(b) Suppose E_1 and E_2 are both Lebesgue measurable. By the direction definition of Lebesgue measurability (via open sets or G_δ approximations), show that the tensor product $E_1 \times E_2$ is also Lebesgue measurable in $\mathbb{R}^n \times \mathbb{R}^m$.

(c) Following (b), show that $|E_1 \times E_2| = |E_1| \times |E_2|$ By Tonelli's Theorem.

(4) (Tensor products of functions; 10pts) Let $f(x, y) = g(x)h(y)$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and $g, h \ge 0$. Suppose in terms of outer measures,

$$|g > 0|_e > 0 \quad \text{and} \quad |h > 0|_e > 0, \tag{1}$$

where $g > 0 = \{x \in \mathbb{R}^n \mid g(x) > 0\}$.

(a) Show that $f(x, y)$ is Lebesgue measurable in $\mathbb{R}^n \times \mathbb{R}^m$ if and only if $g(x)$ and $h(y)$ are measurable in $\mathbb{R}^n$ and $\mathbb{R}^m$.

(b) Show that in one direction of the "if and only if," the condition in Eqn. (1) is necessary.

(5) (Convolution; 40pts) Let $f(x), g(x) \ge 0$ be two measurable functions on $\mathbb{R}^n$. Their convolution is defined by:

$$f * g(x) = \int_{\mathbb{R}^n} f(x - t)g(t)dt \ge 0,$$

which could be ∞ for some x's.

(a) Apply Tonelli's Theorem to show that $f * g$ is measurable in $\mathbb{R}^n$.

(b) Show that $\displaystyle\int_{\mathbb{R}^n} f * g(x)dx = \left(\int_{\mathbb{R}^n} f(x)dx\right)\left(\int_{\mathbb{R}^n} g(x)dx\right).$

(c) Show that if $f, g \in L(\mathbb{R}^n)$, then $f * g \in L(\mathbb{R}^n)$ as well.

(d) (Why (c) is a non-trivial result) On $\mathbb{R}^1$, define

$$f(x) = \sum_{n=1}^{\infty} n\, 1_{[n, n+n^{-3})}(|x|), \quad g(x) = f(x).$$

(In signal processing, f and g are called spike trains.) Show that (d.1) $f, g \in L(\mathbb{R})$, (d.2) $f * g(0) = \infty$, and (d.3) in fact, $f * g(x) = \infty$ **if and only if** $x \in \mathbb{Z}$.

(6) (Fundamental Theorem of Fourier Analysis; 10pts) The *Fourier transform* of $f \in L^(\mathbb{R}^1)$ is define by

$$\hat{f}(\omega) = \int_{\mathbb{R}^1} f(x)e^{-i\omega x}dx, \quad \omega \in \mathbb{R}^1.$$

(a) Show that $\hat{f} \in L^\infty(\mathbb{R}^1)$, i.e., $|\hat{f}(\omega)| \le C$ for some fixed finite upper bound C for all ω.

(b) By Fubini's Theorem, show that if $f, g \in L(\mathbb{R}^1)$, then $\widehat{f * g} = \widehat{f} \times \widehat{g}$.