Problem Set 11, Math8602-Real Analysis

Assigned on Wed, Feb 9, 2005; Due on Monday, Feb 21, 2005

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Hardy-Littlewood's maximal function (HL-MF); 20pts) Recall that for any $f \in L^+_{\rm loc}(\mathbb{R}^n)$, the HL-MF is defined by $f^*(x) = \sup_{Q\sim x}\langle f\rangle_Q$, where $Q \sim x$ denotes all cubes centered at x. In the lecture, it is proven that for any *positive* and *bounded* measurable set E, there exist $A = A(E), C = C(E), R = R(E) > 0$ such that $A|x|^{-n} \le 1^*_E(x) \le C|x|^{-n}$ for all $|x| \ge R$.

(1.1) Show that for any constant function a, $a^* \equiv a$; and that if $f \equiv a$ in a neighborhood of x_0, then $f^*(x_0) \ge a$.

(1.2) Suppose $f \in L^+_{\rm loc}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f > 0$. Show that $f^*(x) \ge A|x|^{-n}$ for all $|x| \ge R$ for some $A, R > 0$.

(1.3) Suppose $f \in L^+_{\rm loc} \setminus L^+$, i.e., $\int_{\mathbb{R}^n} f = \infty$. <u>*Construct*</u> an example to show that it is possible that $|x|^n f^*(x) \to \infty$ as $|x| \to \infty$. (Thus, C does not exist for the upper bound.)

(1.4) Suppose $f \in L^+$. <u>*Construct*</u> an example to show that it is *still* possible that $\overline{\lim}_{x\to\infty}|x|^n f^*(x) = \infty$.

(1.5) Finally, suppose $f \in L^+$, $\int_{\mathbb{R}^n} f > 0$ and f is *compactly supported*. Show that there exist $A = A(f), C = C(f), R = R(f) > 0$ such that $A|x|^{-n} \le f^*(x) \le C|x|^{-n}$ for all $|x| \ge R$.

(2) (Truncated HL-MF; 20pts) Let $f \in L^+_{\rm loc}(\mathbb{R}^n)$. Fixing any radius $R > 0$, we define the *truncated* HL-maximal function by $f^*_R(x) = \sup_{Q\sim x, r(Q)\le R}\langle f\rangle_Q$, where <u>$r(Q)$ denotes the ∞-norm radius when a cube is treated as a ball $B^\infty(x, r)$ under the ∞-norm.</u>

(2.1) Show that if $f(x)$ is compactly supported, so is $f^*_R(x)$.

(2.2) Suppose $f \in L^+$. Show that f^*_R is *weak* L^+. In fact, show that there exists a universal finite constant $C = C(n)$, such that for any $f \in L^+$ and $\alpha > 0$, (i.e., *Hardy-Littlewood lemma* for f^*_R)

$$|\{f^*_R > \alpha\}| \le \frac{C}{\alpha}\int_{\mathbb{R}^n} f. \qquad \text{(Hint: by \underline{Vitali's Covering Lemma})}$$

(2.3) For a signed function f, we similarly define $f^*_R = |f|^*_R$. By (2.2) show that if $f_n \to f$ in $L(\mathbb{R}^n)$, then $(f - f_n)^*_R \to 0$ *in measure*. In particular, there is a subsequence (n_k) such that $(f - f_{n_k})^*_R \to 0$ *almost everywhere* as $k \to \infty$.

(3) (Weak l^1, very important in modern *Wavelets Theory*; 10pts) Like L^1 functions, an indexed sequence (wavelets coefficients, say) $c = (c_n)_{n=1}^\infty$ is said to be l^1 if $\sum_{n=1}^\infty |c_n| < \infty$. On the other hand, c is said to be *weak* l^1 if for any $\alpha > 0$, <u>$\#\{c_n : |c_n| > \alpha\} \le \frac{A}{\alpha}$</u>, for some fixed finite $A > 0$.

(3.1) By the discrete version of <u>Tchebyshev's inequality</u>, show that l^1 implies weak l^1.

(3.2) Suppose c is weak l^1. Show that the *nonzero* elements of c can be ordered by

$$|c_{n_1}| \ge |c_{n_2}| \ge |c_{n_3}| \ge \cdots .$$

(On the other hand, for example, the sequence $c = (1 - n^{-1})$ cannot be ordered in this way.)

(3.3) Show that there exists a finite $A > 0$, such that $|c_{n_k}| \le \dfrac{A}{k}$, $k = 1 : \infty$.

(4) (Generalized HL-MF/HL's lemma; 20pts) Let μ be a general *finite* Borel measure in $\mathbb{R}^n$ (i.e., supported on the Borel σ-algebra and $\mu(\mathbb{R}^n) < \infty$). Define its maximal function ρ^* by $\rho^*(x) = \sup_{Q\sim x}\dfrac{\mu(Q)}{|Q|}$ with cubes.

(4.1) Let $\mu = \delta$ be <u>Dirac's delta measure</u> so that $\delta(E) = 0$ if $0 \notin E$ and $\delta(E) = 1$ otherwise. Explicitly compute its maximal function $\rho^*(x)$. At which point ρ^* blows up?

(4.2) Suppose one knows that $\rho^*(x) \to 0$ as $x \to \infty$. Show (by *Vitali's Covering Lemma*) that Hardy-Littlewood's weak L^1 result still holds for ρ^*:

$$|\{\rho^* > \alpha\}|_e \le \frac{C}{\alpha}\mu(\mathbb{R}^n), \quad \text{for some fixed } C = C(n) \text{ and all } \alpha > 0.$$

For Dirac's delta, what is the optimal (i.e., smallest) $C = C(n)$?

(5) (How far to go beyond Vitali's Covering Lemma (VCL) (or *Inflated Covering*); 20pts) **Some notations and terminologies from the lecture.** A cube cover $\mathcal{K}$ of any set $E \subseteq \mathbb{R}^n$ is a nonempty collection of cubes $\mathcal{K} = \{Q's\}$ such that $E \subseteq \cup_{Q\in\mathcal{K}} Q$. A sub-collection $\mathcal{D} \subseteq \mathcal{K}$ is a disjoint class if any two distinct cubes in $\mathcal{D}$ must also be disjoint. Any cube Q can also be treated as an ∞-norm ball $Q = B^\infty(\boldsymbol{c}, r)$ with center $\boldsymbol{c}(Q)$ and radius $r(Q)$. A cube cover $\mathcal{K}$ is said to be bounded if $r(Q) < R$ for any $Q \in \mathcal{K}$ and some finite $R > 0$. We also define $\lambda[Q] = B^\infty(\boldsymbol{c}(Q), \lambda r(Q))$ for any $\lambda > 0$. Then Vitali's Covering Lemma says: *Any bounded cube cover $\mathcal{K}$ of a set E has a disjoint class $\mathcal{D}$ such that,* $E \subseteq \bigcup_{Q\in\mathcal{D}} 5[Q]$.

(5.1) Construct an example to show that the condition of *boundedness* is necessary.

(5.2) Show that the inflation number 5 can be replaced by any $\rho > 3$. (Hint: by following the spirit of the original proof but with a natural modification.)

(5.3) Show that if $\mathcal{K}$ is a bounded cube cover of E, then there exists a disjoint class $\mathcal{D}$ such that

$$|E|_e \le 5^n \sum_{Q\in\mathcal{D}} |Q|,$$

from which to further show that, for any fixed number $\beta \in (0, 5^{-n})$, as long as $|E|_e < \infty$, one can always find a *finite* disjoint class $\mathcal{D} \subseteq \mathcal{K}$ with $\#\mathcal{D} < \infty$, such that

$$|E|_e \le \beta^{-1} \sum_{Q\in\mathcal{D}} |Q|.$$

(5.4) Show that this (last) result (i.e., when $|E|_e < \infty$) is in fact true even when $\mathcal{K}$ is unbounded.

(5.5) Finally, show that in Vitali's Covering Lemma, the cube class can be replaced by any class of p-balls with $p \in [1, \infty]$. That is $\mathcal{K} = \{B's\}$ with

$$B = B^p(\boldsymbol{c}, r) = \{x \in \mathbb{R}^n : \ |x - \boldsymbol{c}|_p \le r\},$$

under the p-norm. The rest are unchanged, including the inflation number 5.

(6) (Lebesgue's Differentiation Theorem; 10pts) No longer than a page, clearly outline all the major steps in proving Lebesgue's Differentiation Theorem, and have an integrated picture of this most technical chapter (so far).

Caution. *Since most problems have been created independently by the instructor (based on the existing literature), it could happen sometimes that a problem is ill-defined or a result is wrong. If you suspect that is the case, please discuss it with the instructor.*