Problem Set 12, Math8602-Real Analysis

Assigned on Friday, Feb 25, 2005; Due on Monday, March 7, 2005

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Convex functions; 20pts) $f(x)$ is convex on (a, b).

(1.a) Suppose $a < x_0 < x_1 < b$, and $f'(x_0) = f'(x_1)$. Show that f must be *linear* on $[x_0, x_1]$. (In particular, no kinks could lie in between.)

(1.b) Show that for any $x \in (a, b)$, one has $D^+ f(x) = Df(x^+)$ and $D^- f(x) = Df(x^-)$. (Hint: Absolute continuity and the fundamental theorem.)

(1.c) Show that if $f'(x) = Df(x)$ exists for *every* $x \in (a, b)$, it must be a *continuous* function on (a, b).

(2) (Classical Legendre transform of convex functions; 20pts) Assume $f(x)$ is *strictly* convex on (a, b), and $f'(x)$ exists everywhere.

(2.a) Let $p = f'(x)$, and $A = f'(a^+)$ and $B = f'(b^-)$ (A and B could be $\pm\infty$). Show that $x \leftrightarrow p$ is a one-to-one map between (a, b) and (A, B), and *order preserving* (i.e., monotone).

(2.b) Let $x = x(p)$ and $p = p(x)$ under the above correspondence. Define the Legendre transform $g(p) = xp - f(x) = x(p)p - f(x(p))$ for $p \in (A, B)$. Show that $x(p) = g'(p)$. In particular, show that $g(p)$ is *strictly* convex on (A, B). (Thus (x, f) and (p, g) are beautifully symmetric. In thermodynamics, x and p are called conjugate or dual variables.)

(2.c) For $f = x^2/2$ and $f = x^4/4$ on $\mathbb{R}$, compute their Legendre transforms separately.

(3) (Subgradients and subdifferential; 30pts) Let $f(x)$ be a general convex function on (a, b). At any x, for any slope p, define $L_p(y) = f(x) + p(y - x)$ to be the linear line passing through $(x, f(x))$.

(3.a) A subgradient p of f at a given spot x is a slope such that $f(y) \geq L_p(y)$ for all $y \in (a, b)$ (i.e., the graph of f only touches L_p from above at the given point). Suppose $f'(x)$ exists. Show that a subgradient at x must be identical to the ordinary gradient $p = f'(x)$.

(3.b) Generally, at any given x, let $\partial f(x)$ denote the collection of *all* subgradients of f at x. Then $\partial f(x)$ is called the subdifferential of f at x. For $f(x) = |x|$, compute $\partial f(0)$.

(3.c) Generally Show that $\partial f(x) = [D^- f(x), D^+ f(x)]$. (In particular, $\partial f(x)$ is non-empty everywhere.)

(3.d) Show that f is convex on (a, b) *if and only if* for any $x \in (a, b)$, $\partial f(x)$ is non-empty.

(3.e) Suppose that $x_0 < x_1$, and $\partial f(x_0) \cap \partial f(x_1)$ is non-empty. Show that f must be linear on $[x_0, x_1]$.

(3.f) Show that $(f'(a^+),\ f'(b^-)) \subseteq \cup_{x\in\mathbb{R}} \partial f(x) \subseteq [f'(a^+),\ f'(b^-)]$.

(3.g) For any two sets D and E, we define $D \leq E$ if and only if for any $d \in D$ and $e \in E$ one has $d \leq e$. Show that convexity implies that for any $x < y$ (in (a, b)), $\partial f(x) \leq \partial f(y)$.

(4) (Generalized Legendre transform; 30pts) Assume that $f(x)$ is convex on (a, b). For any $p \in (A, B)$ with $A = f'(a^+)$ and $B = f'(b^-)$, the generalized Legendre transform is defined by $g(p) = \sup_{x\in\mathbb{R}} (px - f(x))$. (In particular, $g(p) + f(x) \geq px$ for any p and x.)

(4.a) Suppose that $p = f'(x_0)$ for some x_0. Show that $g(p) = px_0 - f(x_0)$.

(4.b) More generally, show that $p \in \partial f(x_0)$ *if and only if* $g(p) = px_0 - f(x_0)$.

(4.c) Similarly show that $x_0 \in \partial g(p)$ *if and only if* $f(x_0) = px_0 - g(p)$.

(4.d) Show that $g(p)$ is also a convex function on (A, B), and that the beautiful duality (generalizing (2.b)) still holds: $p \in \partial f(x_0)$ if and only if $x_0 \in \partial g(p)$. [Hint: using (3.d).]

(5) (Integration by parts for absolutely continuous (A.C.) functions; 10pts)

(5.a) By the definition of A.C., show that if $g(x)$ and $f(x)$ are both A.C. on (a, b), then the product $h(x) = g(x)f(x)$ must be A.C. as well.

(5.b) Show that the following classical integration-by-parts formula still holds for two A.C. functions f and g:

$$\int_a^b g f' dx = -\int_a^b g' f dx + g(b)f(b) - g(a)f(a).$$

(6) (Monotone BV; 10pts) Let f be a monotone function on $(0,1)$. Then we have shown that f is differentiable almost everywhere. Show that $0 \le \int_0^1 f' \le f(1^-) - f(0^+)$. [In particular, f' is Lebesgue integrable if both $f(1^-)$ and $f(0^+)$ are finite.] Furthermore, verify its validity for the Cantor function $C(x)$. [Hint: By using Fatou's Lemma for the sequence $D_h f(x) = (f(x+h) - f(x))/h$, with proper extension at the ends.]

(7) (Jordan-Lebesgue decomposition for BV; 10pts) Inspired by the Cantor function, we say a function h is singular on $(0,1)$ if $h'(x) = 0$ a.e. Show that for any $f \in \mathrm{BV}(0,1)$, there exists a unique decomposition of $f(x) = g(x) + h(x)$, where h is singular and g is absolutely continuous with $h(0^+) = 0$.

(8) (BV v.s. AC; 10pts) Suppose $f \in \mathrm{BV}(0,1)$, and $V(x) = V(0,x)$ its variation function. Show that f is absolutely continuous if and only if $V(x)$ is absolutely continuous.

Attn: You can select any combination to make a perfect score of 100 points, and you are entitled to work *only* on those problems and ignore the others. For example, combination A (1, 2, 3, 4: 20+20+30+30=100), and combination B (1, 2, 3, 5, 7, 8: 20+20+30+10+10+10=100). We will grade and count any problem you have worked on, however. The highest total score will still be 100, another however.