Problem Set 13, Math8602-Real Analysis

Assigned on Monday, April 4, 2005; Due on Monday, April 18, 2005

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Sequence spaces; 10pts) Show that for any $1 \le p \le q \le \infty$, $l^p \subseteq l^q \subseteq l^\infty$, and that $\|c\|_q \le \|c\|_p$ for any sequence $c = (c_n) \in l^p$. (Hint: $\|c\|_\infty \le \|c\|_p$.) (Warning: results here generally do not hold for L^p.)

(2) (Weak*-precompactness and lower semi-continuity; 20pts) Let B be a real *separable* Banach space and S any *countable* dense set. Let $D = \mathrm{span}_{\mathbb{Q}}\{S\}$ denote the linear space spanned by S using only rational coefficients. Then it is easy to see that D is also dense and countable.

- (2.1) Show that $D^* = B^*$, in terms of the dual spaces.
- (2.2) For any sequence $(f_n) \subseteq B^*$ and $f \in B^*$, f_n's are said to converge to f in the weak*-sense (or $w*$) if for any $b \in B$, one has $\langle f_n\ ,\ b\rangle \to \langle f\ ,\ b\rangle$. Suppose that (f_n) is a **bounded** sequence in B^*. Show that $f_n \overset{w*}{\to} f$ iff $\langle f_n\ ,\ d\rangle \to \langle f\ ,\ d\rangle$ for any $d \in D$.
- (2.3) (Weak*-precompactness) Show that any *bounded* sequence (f_n) in B^* must be weak*-precompact. That is, there must exist a subsequence (f_{n_k}) and some $g \in B^*$, such that $f_{n_k} \overset{w*}{\to} g$ as $k \to \infty$.
- (2.4) (Lower semi-continuity under weak* convergence) Suppose $f_n \overset{w*}{\to} g$ in B^*. Then $\|g\| \le \underline{\lim}_{n\to\infty} \|f_n\|$.

(3) (Duality $(l^p)^* = l^q$ with $p^{-1} + q^{-1} = 1$, $1 \le p < \infty$; 20pts) Define the n-th canonical basis sequence e^n by $e^n_j = \delta_{n,j} = 1$, if $j = n$; 0, otherwise. Let $D = \mathrm{span}\{e^n \mid n = 0 : \infty\}$ and $B = l^p$ with $1 \le p < \infty$.

- (3.1) Show that D is dense in B, but not dense (i.e., only sparse) in l^∞.
- (3.2) Show that for any $f, g \in B^*$, $f = g$ on B if and only if $f = g$ on D.
- (3.3) By Hölder's inequality, show that $l^q \subseteq B^*$ naturally.
- (3.4) For any $f \in B^*$, define $d = d_f = (d_n)$ by $d_n = \langle f\ ,\ e^n\rangle$, $n = 0 : \infty$. Show that $d \in l^q$ and $\|d\|_q \le \|f\|$.
- (3.5) Show that in fact $f = d$ as two elements in B^* and $\|f\| = \|d\|_q$. (Thus $(l^p)^* = l^q$ for $1 \le p < \infty$.)

(4) (Weak-precompactness and Riesz's Representation Theorem; 10pts) By Riesz's Representation Theorem, $(L^p)^* = L^q$, for any $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. In particular, for $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, $(L^p)^* = L^q$ and $(L^q)^* = L^p$. Show that any **bounded** sequence of functions (f_n) in $B = L^p$ $(1 < p < \infty)$ must be weakly precompact. That is, there must exist a subsequence (f_{n_k}) and some $f \in B$, such that for any $g \in B^* = L^q$, $\langle f_{n_k}\ ,\ g\rangle \to \langle f\ ,\ g\rangle$, as $k \to \infty$. (Here $L^p = L^p(E)$ for some measurable set $E \subseteq \mathbb{R}^d$.)

(5) (Energy conservation; From Wheeden-Zygmund; 10pts) Suppose $f_n, f \in L^2(\mathbb{R}^d)$, and $f_n \to f$ **weakly** as $n \to \infty$. In addition, assume that no "energy" is lost during the weak convergence: $\|f_n\| \to \|f\|$ as $n \to \infty$. Show that one must have the **strong** convergence $\|f_n - f\| \to 0$ as $n \to \infty$. (Hint: under any orthonormal basis, the problem is essentially in l^2.)

(6) (p-Lebesgue points; From Wheeden-Zygmund; 10pts) A point $x \in \mathbb{R}^d$ is said to be the p-Lebesgue point of $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ (with $1 \le p < \infty$), if (following the notations of the previous chapter)

$$\lim_{Q\to x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy = 0.$$

Show that for any $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, *almost all* points in $\mathbb{R}^d$ are its p-Lebesgue points. (Hint: basically review and copy the proof for the ordinary case of $p = 1$, i.e., using rational interpolations and L.D.T.)

(7) (Basic concepts in Hilbert spaces; 20pts) Let (ϕ_n) be an orthonormal *system* of a separable Hilbert space H. (**a**) State Bessel's inequality and Parseval's formula; (**b**) State the concepts of "completeness" and "basis." (**c**) Prove the equivalence among *completeness*, *basis*, and *Parseval's formula* (for *any* $f \in H$).

(8) (Bonus (research-kind) problem: Haar's wavelets; 10pts) In our lecture we have shown that the linear span D by the indicator functions of all dyadic boxes is dense in $L^p(\mathbb{R}^d)$ for any $p \in [1, \infty)$. Now focus on $H = L^2(0, 1)$. Define Haar's mother wavelet $\psi = 1_{(0,1/2]}(x) - 1_{(1/2,1]}(x)$, with a cliff at $x = 1/2$, **and** $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ for $j \ge 0$ and $0 \le k \le 2^j - 1$. Show that $\{\phi = 1\} \cup \{\psi_{j,k} \mid j \ge 0, 0 \le k \le 2^j - 1\}$ is an **orthonormal basis** of $L^2(0, 1)$. (Hint: show that $V_J = \mathrm{span}\{1, \psi_{j,k} \mid 0 \le j \le J, 0 \le k \le 2^j - 1\}$ is exactly the linear span by all the dyadic intervals of scale $2^{-(J+1)}$.)