Problem Set 14, Math8602-Real Analysis

Assigned on Wed, April 27, 2005;

"Real Analysis for Real People in the Real World" – Jackie Shen ©

(1) (Absolute Continuity (AC) of Measures: 0-0 versus ϵ-δ) Let μ and ν be two ordinary measures on a measurable space (X, Σ). In the lecture, it is shown that the 0-0 statement for A.C. is equivalent to the ϵ-δ one *if ν is finite*. Construct a counterexample when ν is not finite. That is, $\mu(E) = 0$ always implies $\nu(E) = 0$, but the following ϵ-δ statement becomes invalid - "*For any $\epsilon > 0$, there exists some $\delta > 0$, such that $\mu(E) < \delta$ always implies $\nu(E) < \epsilon$.*" [Hint: construct a crazy ν from some μ.]

(2) (Radon-Nikodym Derivatives: A 1-D Warm-Up) Let $dm = dx$ be the 1-D Lebesgue measure, $d\mu = e^{-x^2}dx$ a (finite) Gaussian measure on $\mathbb{R}$, and $d\nu = e^{-x^2}1_{|x|\leq 1}(x)dx$. (a) Show that $\nu \ll \mu \ll m$. (b) Show that $m \ll \mu$. What is the Radon-Nikodym derivative $dm/d\mu$? Does it belong to $L^1(d\mu)$ or only $L^1_{\text{loc}}(d\mu)$? (c) Show that $\mu \ll \nu$ is false. What is the **Lebesgue-Radon-Nikodym decomposition** of μ with respect to ν: $\mu = \mu_{\|} + \mu_{\perp}$ such that $\mu_{\|} \ll \nu$ and $\mu_{\perp} \perp \nu$?

(3) (Hahn Decomposition) Let ν be a *finite* signed measure on a measurable space (X, Σ). Define

$$M_+ = \sup\{\nu(E) \mid E \in \Sigma\}, \quad M_- = \inf\{\nu(E) \mid E \in \Sigma\}.$$

Apply the Hahn Decomposition Theorem to show that both M_+ and M_- must be *finite*.

(4) (Examples of Singular Measures) Consider two Borel measures in $\mathbb{R}^2$. Let $d\mu = dxdy$ be the 2-D Lebesgue measure, and $d\nu(B) = \text{length}(B \cap C)$ for any Borel set $B \subseteq \mathbb{R}^2$, where C denotes the **unit circle** centered at $(0, 0)$. (The length measure on C is a special case of **Haar measures on topological groups**. We can however just understand it in the intuitive way, namely, a "wrapped-around" 1-D Lebesgue measure. Google to find more about Haar measures.) Show that μ and ν are singular to each other.

(5) (Singular Measures from BV) Let f be a right-continuous BV function on $(0, 1)$. We have shown in the previous chapters that f has a unique **Jordan decomposition**: $f = g + h$ where, (1) g is *absolutely continuous*, while h is *singular* (i.e., $h' = 0$, a.e. on (0,1)) and $h(0^+) = 0$. (Assume h is non-decreasing to make life easier.) For any interval $(a, b]$ in $(0, 1)$, define

$$\mu(a, b] = \int_a^b g'(x)dx, \qquad \nu(a, b] = h(b) - h(a).$$

Both μ and ν can be naturally extended to signed (Stieltjes) measures on (0, 1). Show that they are *singular* to each other. What is the Radon-Nikodym derivative $d\mu/dx$? **Vitali's covering lemma helps.**

(6) (Total Variations (TV) of Signed Measures and Test Functions) For a signed measure ν on (X, Σ), let $\nu = \nu^+ - \nu^-$ denote its unique **Jordan decomposition** so that $\nu^\pm$ are ordinary (i.e., nonnegative) measures and are *singular* to each other. The TV measure $|\nu|$ of ν is then defined by $|\nu| = \nu^+ + \nu^-$. Show that for any measurable set $E \in \Sigma$,

$$|\nu|(E) = \sup_{\|\phi\|_\infty \leq 1} \int_E \phi(x)d\nu, \quad \text{with measurable test signals } \phi\text{'s}.$$

Such characterization via test functions ϕ's is extremely useful in applications.

(7) (Lebesgue-Radon-Nikodym (LRN) Decomposition Generalizes Linear Algebra, *in some sense*) LRN Theorem says, given any σ-finite signed measure ν and ordinary measure μ (on some measurable space (X, Σ)), we can decompose $\nu = \nu_{\|} + \nu_{\perp}$ such that $\nu_{\|} \ll \mu$ (AC component) and $\nu_{\perp} \perp \mu$ (singular component). Let us see in what sense the linear algebra notions of "parallel" and "perpendicular" may be helpful.

(a) On $(0, 1)$, consider $d\mu = 1 \cdot dx$, and $d\nu = x \cdot dx$. Since 1 is a unit vector in $L^2(0, 1)$, the function x can be decomposed into $x = x_{\|} + x_{\perp}$, with $x_{\|} = \langle x\ ,\ 1\rangle 1 = 1/2$, and thus $x_{\perp} = x - 1/2$. But show that $\nu_{\|} = \nu$ and $\nu_{\perp} = 0$. Thus LRN decomposition is not a linear algebra one.

(b) Now on $(0, 1)$, define $d\mu = 1_{x>1/2}(x) \cdot dx = p(x)dx$ and $d\nu = 1_{x\leq 1/2}(x) \cdot dx = q(x)dx$. Show that this time $\nu_{\|} = 0$ and $\nu_{\perp} = \nu$, which is consistent with the linear algebra decomposition since $\langle p\ ,\ q\rangle = 0$.