The Gelfand-Levitan method,
in the the vector-valued case, I

Max Jodeit, Jr. and B. M. Levitan
School of Mathematics
University of Minnesota

Introduction

This work is for application to certain Sturm-Liouville problems, especially the isospectrality problem for vector-valued two-point problems on a bounded interval. We want to use transmutation operators of the form $I+\mathcal{K}$ to attack the isospectrality problem, where $\mathcal{K}$ is a Volterra integral operator. The kernel for the operator $\mathcal{K}$ arises as the solution of a transformation equation. A transformation equation has the form

$$\mathcal{K}(x,\,y)+\mathcal{F}(x,\,y)+\int_0^x \mathcal{K}(x,\,t)\mathcal{F}(t,\,y)\,dt=0,\ \ 0\le y\le x\le \pi.$$

In this equation, $\mathcal{F}(x,\,y)$ is a known function defined on the square $[0,\,\pi]^2$. The transmutation operator then transforms eigenfunctions of an original, or "unperturbed," Sturm-Liouville problem into eigenfunctions of a new, or "perturbed," problem, whose potential is expressible in terms of $\mathcal{K}(x,\,x)$.

The "known function" has the form $\mathcal{F}(x,\,y)=\sum_{n=0}^{\infty} c_n\,\varphi_n^o(x)\,\varphi_n^o(y)^T$. The functions $\varphi_n^o(x)$ are from an orthonormal basis of vector-valued L^2 that consists of the eigenfunctions of a Sturm-Liouville problem with Neumann-type boundary conditions. The scalars c_n are the terms of an absolutely convergent series that satisfies the condition $1+c_n>0$ for all integers $n\ge 0$.

In this paper, we concentrate on some functional-analytic features of the transformation equation that make no direct use of the fact that the functions $\varphi_n^o(x)$ are eigenfunctions of a Sturm-Liouville problem. The orthonormal bases $\{\varphi_n^o(x)\}$ are not arbitrary, however. We do use some special properties enjoyed by bases of eigenfunctions of one of our Sturm-Liouville problems, namely that they consist of continuous, uniformly bounded vector-valued functions and that they comprise a set that is linearly independent on intervals $[0, x]$ when $0 < x < \pi$. We thus use the transformation equation to pass from certain special sequences $\{c_n\}$ in ℓ^1 to a continuous matrix-valued kernel $\mathcal{K}(x, y)$ defined on the triangle $0 \le y \le x \le \pi$, and the resulting Volterra operator on $L^2([0, \pi]; \mathbb{R}^d)$, the Hilbert space of vector-valued functions defined on $[0, \pi]$. Sections 1 – 4 are devoted to proving the existence and continuity of $\mathcal{K}$, and to developing some other material we will need later.

In Section 5 we introduce the transformed functions $\varphi_n(x) = \varphi_n^o(x) + \int_0^x \mathcal{K}(x, y)\varphi_n^o(y)\, dy$. They are needed in most of the subsequent sections. In the context of Sturm-Liouville problems, they become eigenfunctions of a "perturbed" problem.

When the functions $\{\varphi_n^o(x)\}$ come from one of our Sturm-Liouville problems, the "perturbed" Sturm-Liouville problem has a potential $Q(x)$ that is expressible in terms of $\mathcal{K}(x, x)$. To be sure that the "perturbed" problem is self-adjoint, we need to know that $\mathcal{K}(x, x)$ is symmetric for each $x \in [0, \pi]$. This is proved in Section 6. The proof has two main steps. First, the proof is done when only a finite number of the basis functions $\varphi_n^o(x)$ appear in the transformation equation. We call this the "finite" case. The linear independence of the basis functions on intervals $[0, x]$ is crucial in this step. The second step is proof of convergence as the number of basis functions used tends to infinity. The uniform boundedness is used here.

Section 7 contains the derivation of a formula for $\mathcal{K}(x, y)$ in the "finite" case that we use in Section 8 to prove a "closed-form" formula for the inverse of $I + \mathcal{K}$. The formula is not new; it is the vector analog of the formula in the scalar case, which was originally proved with the help of a Goursat problem, in the context of Sturm-Liouville problems. **(Reference!)** The formula still holds, without the differential equation, and the inverse has the form $I + \mathcal{L}$, where $\mathcal{L}$, like $\mathcal{K}$, is a Volterra integral operator.

In Section 9 we observe that, in a sense, the transformation equation can be extended to the whole square $[0, \pi]^2$. This gives a kernel $\mathcal{K}_{\#}(x, y)$ defined on the whole square. The dual kernel, $\mathcal{K}_{\#}(y, x)^T$, is the (negative of) the (similarly done) extension, $\mathcal{L}_{\#}(x, y)$, of the kernel $\mathcal{L}(x, y)$ for the inverse, studied in Section 8. That is, $\mathcal{L}_{\#}(x, y) = -\mathcal{K}_{\#}(y, x)^T$. This observation is used in Section 10 to prove the orthogonality of the transformed functions, and write down a Parseval equation for them.

In Section 10 we show that the transformed functions $\varphi_n(x) = \varphi_n^o(x) + \int_0^x \mathcal{K}(x, y)\varphi_n^o(y)\, dy$, introduced in Section 5, comprise an orthogonal set whose span is dense. In the context of Sturm-Liouville problems, they yield a complete system of eigenfunctions for a "perturbation" of the original Sturm-Liouville problem. In the present paper we assume about $\{\varphi_n^o(x)\}$ only the properties that we mentioned before: continuity, uniform boundedness, linear independence on intervals $[0, x]$, for $0 < x < \pi$. Nevertheless, the set of transformed functions is still orthogonal.

When the transformed functions are normalized, one can check routinely that they satisfy the hypotheses we assumed about the original orthonormal basis. In Section 11 we find the transformation equation for the new basis that one uses to recover the original one in the same manner.

§1 Preliminaries

In this paper we will suppose that $\{\varphi_n^o(x)\}_{n\geq 0}$ is an orthonormal basis of the Hilbert space $L^2([0,\,\pi];\,\mathbb{R}^d)$ of square-integrable d-dimensional column-vector-valued functions. We will also suppose that the functions $\varphi_n^o(x)$ are continuous and pointwise uniformly bounded, namely that there is a constant A, independent of n, such that $|\varphi_n^o(x)| \leq A$ for $0 \leq x \leq \pi$. Here, $|\varphi_n^o(x)|$ is the Euclidean norm of $\varphi_n^o(x)$. In addition, we assume that the set of functions $\{\varphi_n^o(x)\}$ is linearly independent on $[0,\,x]$ for each $x \in (0,\pi]$.

A compact "diagonal" operator

We construct a $d \times d$ matrix-valued kernel for an integral operator $\mathcal{F}$, as follows. Let c_n, $n = 0, 1, 2, \ldots$ be real numbers, with $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the function

$$\mathcal{F}(x,\,y) := \sum_{n=0}^{\infty} c_n\, \varphi_n^o(x)\, \varphi_n^o(y)^T \tag{1.1}$$

is continuous on the square $0 \leq x \leq \pi$, $0 \leq y \leq \pi$. We define an integral operator $\mathcal{F}$ by

$$\mathcal{F}f(x) := \int_0^{\pi} \mathcal{F}(x,\,y) f(y)\, dy, \tag{1.2}$$

so $\mathcal{F}$ is a compact operator on $L^2([0,\,\pi];\,\mathbb{R}^d)$, and on $C([0,\,\pi];\,\mathbb{R}^d)$. We will also regard $\mathcal{F}$ as an operator on the spaces $L^2([0,\,\pi];\,\mathbb{R}^{d\times d})$ and $C([0,\,\pi];\,\mathbb{R}^{d\times d})$, which consist of $d \times d$ matrix-valued functions $f(y)$, where $|f(y)|$ now denotes the operator norm of the matrix $f(y)$ acting on $\mathbb{R}^d$, equipped with the Euclidean norm. If no confusion is likely,

$\|f\|$ may denote one of the norms $\|f\|_2 := \sqrt{\int_0^\pi |f(y)|^2 dy}$, or $\|f\|_\infty := \max_{0\le y\le \pi} |f(y)|$, depending on which of our spaces, such as $L^2([0,\,\pi];\,\mathbb{R}^{d\times d})$ or $C([0,\,\pi];\,\mathbb{R}^{d\times d})$, is in use. Similar definitions of norms apply to the spaces $L^2([0,\,\pi];\,\mathbb{R}^{d})$ and $C([0,\,\pi];\,\mathbb{R}^{d})$. The norm notation for operators such as $\mathcal{F}$ will also be $\|\mathcal{F}\|$, and the context in which the operator is used will usually suggest how the operator's norm is defined.

§2 The transformation equation

The following integral equation for an unknown $d \times d$ matrix-valued function $\mathcal{K}(x,\,y)$ has become known as "the Gelfand-Levitan equation" but we will call it a *transformation equation:*

$$\mathcal{K}(x,\,y) + \mathcal{F}(x,\,y) + \int_0^x \mathcal{K}(x,\,t)\mathcal{F}(t,\,y)\,dt = 0,\ \ 0 \le y \le x \le \pi. \tag{2.1}$$

We intend for the solution $\mathcal{K}(x,\,y)$, if it exists, to become the kernel for a Volterra integral operator $\mathcal{K}$, given by

$$\mathcal{K}f(x) := \int_0^x \mathcal{K}(x,\,y)f(y)\,dy.$$

We first show, in (2.2), that the transformation equation (2.1) has a unique solution $\mathcal{K}(x,\,y)$ under a global condition on the numbers c_n. The proof of the existence of $\mathcal{K}(x,\,y)$ for $y \in [0,\,x]$ is a straightforward adaptation, to the vector-valued case, of the proof of Theorem 1.1 in [JL]. We give this adaptation here because we want to use parts of it later, and because we want to work with the transposed version of the transformation equation. In (4.1) we will show that $\mathcal{K}(x,\,y)$ is continuous on the triangle $0 \le y \le x \le \pi$. We can say that the Volterra integral operator $\mathcal{K}$ is *induced* by the operator $\mathcal{F}$.

(2.2) **Theorem** *Suppose that, for all* $n \geq 0$, *we have*

$$1 + c_n > 0. \tag{2.3}$$

Then the integral equation (2.1) *has a unique solution* $\mathcal{K}(x,\, y)$, *for* $0 \leq y \leq x$, *for every* $x \in (0,\, \pi]$.

Proof: The proof may be easier to read if we first transpose the matrices in the transformation equation. Once that is done, we can prove the theorem "column-by-column." Thus we consider the transformation equation in the following form, exploiting the symmetry relation $\mathcal{F}(x,\, y)^T = \mathcal{F}(y,\, x)$:

$$\mathcal{K}(x,\, y)^T + \mathcal{F}(y,\, x) + \int_0^x \mathcal{F}(y,\, t)\mathcal{K}(x,\, t)^T \, dt = 0, \quad 0 \leq y \leq x \leq \pi. \tag{2.4}$$

Since $\mathcal{F}$, as defined in (1.2), is a compact operator, it is enough to prove that the trivial solution is the only solution of the homogeneous column-vector version of (2.4), namely the equation that follows, for a function $h_x(y) \in L^2(\,[0,\, \pi];\, \mathbb{R}^d)$:

$$h_x(y) + \int_0^x \mathcal{F}(y,t)h_x(t)\, dt = 0, \quad 0 \leq y \leq x. \tag{2.5}$$

Thus, suppose $h_x(y)$ is a solution of (2.5). We extend $h_x(y)$ to $[0,\, \pi]$ by setting $h_x(y) \equiv 0$ in $(x,\, \pi]$. Then we have, in the L^2-sense,

$$h_x(y) = \sum_{n=0}^{\infty} \left(\int_0^{\pi} \varphi_n^o(t)^T h_x(t)\, dt \right) \varphi_n^o(y),$$

since the system $\{\varphi_n^o(x)\}$ is complete.

Let us multiply the equation (2.5) on the left by $h_x(y)^T$, integrate in y from 0 to x, or from 0 to π, which amounts to the same thing:

$$\int_0^{\pi} |h_x(y)|^2 \, dy + \int_0^{\pi} \int_0^{\pi} h_x(y)^T \mathcal{F}(y,t) h_x(t)\, dt\, dy = 0. \tag{2.6}$$

From Parseval's equation, we have

$$\int_0^\pi |h_x(y)|^2\,dy = \sum_n \left(\int_0^\pi \varphi_n^o(t)^T h_x(t)\,dt \right)^2 . \tag{2.7}$$

When we substitute (2.7) and (1.1) into (2.6), and interchange the sum and integrals, we find that

$$\begin{aligned} 0 &= \sum_n \left(\int_0^\pi \varphi_n^o(t)^T h_x(t)\,dt \right)^2 + \int_0^\pi \int_0^\pi h_x(y)^T \left(\sum_{n=0}^\infty c_n\, \varphi_n^o(y)\, \varphi_n^o(t)^T \right) h_x(t)\,dt\,dy \\ &= \sum_n (1+c_n) \left(\int_0^\pi \varphi_n^o(t)^T h_x(t)\,dt \right)^2 . \end{aligned}$$

From this last equation and condition (2.3) it follows that

$$\int_0^\pi \varphi_n^o(y)^T h_x(y)\,dy = 0 \ \text{ for all } \ n.$$

Since the system $\{\varphi_n^o(x)\}$ is complete, and $h_x(y)$ is necessarily continuous on $[0,\, x]$, we conclude that $h_x(y) \equiv 0$. The theorem follows, with the help of Fredholm's theory.

Remark Only the orthonormality of $\{\varphi_n^o(x)\}$ was used.

§3 For convenience: "scaled" versions of the operators $\mathcal{F}_x$

This section is devoted to norm estimates, in more than one context, for the inverse of the operator $I + \mathcal{F}$ that appeared in §2. These estimates will be used to show that $\mathcal{K}(x,\, y)$ is continuous in x and y, that $\mathcal{K}(x,\, x)$ is symmetric, and for deriving a "closed-form formula" for the inverse of the operator $I + \mathcal{K}$. In the proof of Theorem (2.2) we worked with the restrictions $\mathcal{F}_x$ of the operator $\mathcal{F}$ to the spaces $L^2([0,\, x];\, \mathbb{R}^{d\times d})$, $0 < x \le \pi$. It will be convenient to "make a copy" of $\mathcal{F}_x$ and use it on $L^2([0,\, tx];\, \mathbb{R}^{d\times d})$, where t is close to one. We will use the "scaling" operators S_t that stretch or shrink the domain of a function to accomplish this.

For $0 < x \le \pi$, we define integral operators $\mathcal{F}_x$ by

$$\mathcal{F}_x f(y) := \int_0^x \mathcal{F}(y,\, s) f(s)\, ds, \quad 0 \le y \le x, \quad \text{where} \;\; f \in L^2([0,\, x];\, \mathbb{R}^{d\times d}). \tag{3.1}$$

By the proof of (2.2), each of the operators $I + \mathcal{F}_x$ is invertible on its domain, namely $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. We will also use the fact that each of the operators $I + \mathcal{F}_x$ is invertible on $C([0,\, x];\, \mathbb{R}^{d\times d})$ as well. The following theorem will be a useful lemma for us.

(3.2) **Theorem:** *For* $0 < x \le \pi$, *let us consider the operator* $\mathcal{W}_x := I + \mathcal{F}_x$, *as an operator on* $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. *We set* $\nu(x) := \|\mathcal{W}_x^{-1}\|$, $0 < x \le \pi$, *and define* $\nu(0) := 1$. *Then* $\nu(x)$ *is continuous on* $[0,\, \pi]$, *and thus is bounded below by a positive quantity.*

Proof: We need information about the norm of the inverse of $\mathcal{W}_x$, as a function of x, so we will make changes of variable that allow us to compare $\mathcal{W}_x$ and $\mathcal{W}_{tx}$, even though they are defined on different spaces. We will think of t as being close to 1, and think of x as fixed.

For $t > 0$, we define the "scaling operator" S_t on $L^2([0,\, \pi];\, \mathbb{R}^{d\times d})$ by

$$S_t f(y) := \begin{cases} f(ty), & \text{if } 0 \le y \le \min(\pi/t,\, \pi); \\ 0, & \text{if } y > \min(\pi/t,\, \pi). \end{cases} \tag{3.3}$$

We will use the restrictions of these operators to the spaces $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. We will always regard S_t as an invertible operator from $L^2([0,\, tx];\, \mathbb{R}^{d\times d})$ to $L^2([0,\, x];\, \mathbb{R}^{d\times d})$, with norm $1/\sqrt{t}$, whose inverse $S_{1/t}$, with norm $\sqrt{t}$, maps $L^2([0,\, tx];\, \mathbb{R}^{d\times d})$ onto $L^2([0,\, x];\, \mathbb{R}^{d\times d})$, provided that $0 < t \le \pi/x$. Proofs of these statements are immediate.

We need to define operators induced by applying S_t to $\mathcal{F}_{tx}$. These will be the "scaled operators," $\mathcal{F}_{*t}$, that are operators on $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. From the equations

$$S_t \mathcal{F}_{tx} f(y) = \mathcal{F}_{tx} f(ty) = \int_0^{tx} \mathcal{F}(ty,\, s) f(s)\, ds = \int_0^x t\mathcal{F}(ty,\, ts) f(ts)\, ds, \tag{3.4}$$

we see that it is reasonable to define $\mathcal{F}_{*t}$ on $L^2([0,\, x];\, \mathbb{R}^{d\times d})$ by

$$\mathcal{F}_{*t}g(y) := \int_0^x t\mathcal{F}(ty,\, ts)g(s)\, ds, \quad 0 \le y \le x, \quad \text{where } \ g \in L^2([0,\, x];\, \mathbb{R}^{d\times d}). \tag{3.5}$$

Then, in terms of $\mathcal{F}_{*t}$, (3.4) becomes

$$S_t\mathcal{F}_{tx}f(y) = \mathcal{F}_{tx}f(ty) = \mathcal{F}_{*t}S_tf(y), \quad 0 \le y \le x. \tag{3.6}$$

Consequently we have

$$S_t(I + \mathcal{F}_{tx}) = (I + \mathcal{F}_{*t})S_t \ \text{ or } \ S_t\mathcal{W}_{tx} = \mathcal{W}_{*t}S_t, \tag{3.7}$$

Where $\mathcal{W}_{*t} := I + \mathcal{F}_{*t}$ is an operator on $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. Since $\mathcal{W}_{tx} = S_{1/t}\mathcal{W}_{*t}S_t$, and since the norms of S_t and $S_{1/t}$ are reciprocals of each other, $\|\mathcal{W}_{tx}\| = \|\mathcal{W}_{*t}\|$. To complete the proof that $\|\mathcal{W}_x^{-1}\|$ is continuous for $x \in (0,\, \pi]$, it is enough to show that $\mathcal{W}_{*t}$ is continuous, as a function of t in $(0, \pi/x]$, into the space of bounded operators on $L^2([0,\, x];\, \mathbb{R}^{d\times d})$, for it then follows that $\mathcal{W}_x^{-1}$ is continuous there also. The continuity of $\mathcal{W}_{*t}$ follows from the continuity of the matrix-valued function $\mathcal{F}(x,\, y)$. Therefore, by the triangle inequality and the fact that $\|\mathcal{W}_{tx}\| = \|\mathcal{W}_{*t}\|$, we may conclude that $\|\mathcal{W}_x^{-1}\|$ is continuous in x for $0 < x \le \pi$.

Here are details on the continuity of $\mathcal{W}_{*t}$. For a fixed x, and appropriate s and t,

$$(\mathcal{W}_{*t} - \mathcal{W}_{*s})g(y) = \int_0^x (t\mathcal{F}(ty,\, tu) - s\mathcal{F}(sy,\, su))g(u)\, du.$$

The kernel here can be written as

$$(t-s)\mathcal{F}(ty,\, tu) + s(\mathcal{F}(ty,\, tu) - \mathcal{F}(ty,\, su)) + s(\mathcal{F}(ty,\, su) - \mathcal{F}(sy,\, su)).$$

This gives a small norm for the difference, here in terms of the modulus of continuity of $\mathcal{F}(y,\, u)$. This formula can also be used to show continuity under less restrictive conditions on $\mathcal{F}(y,\, u)$, such as uniform continuity in each variable, in the norm in $L^2([0,\, \pi];\, \mathbb{R}^{d\times d})$.

When x is near 0, we can estimate the norm of $(I+\mathcal{F}_x)^{-1} - I$ directly from the series $\sum_{n=0}^{\infty}(-\mathcal{F}_x)^n = (I+\mathcal{F}_x)^{-1}$, since $\|\mathcal{F}_x\| \le \|\mathcal{F}\|_\infty \sqrt{x}$. Then, when we define $\nu(x) = \|\mathcal{W}_x^{-1}\|$, $0 < x \le \pi$, and define $\nu(0) = 1$, as in the statement of our theorem, it follows that $\nu(x)$ is continuous on $[0,\, \pi]$, hence bounded below by a positive quantity. This completes the proof.

Remark: The proof still works in the spaces $C([0,\, x];\, \mathbb{R}^{d\times d})$ in place of the spaces $L^2([0,\, x];\, \mathbb{R}^{d\times d})$. The quantity $\sqrt{x}$ in the norm estimates must then be replaced by x, and the norms of all the relevant S_t become 1. This gives us a Corollary that we will use later.

(3.9) **Corollary:** *For* $0 < x \le \pi$, *let us consider the operator* $\mathcal{W}_x := I + \mathcal{F}_x$ *as an operator on* $C([0,\, x];\, \mathbb{R}^{d\times d})$. *We set* $\mu(x) := \|\mathcal{W}_x^{-1}\|$, $0 < x \le \pi$, *and define* $\mu(0) := 1$. *Then* $\mu(x)$ *is continuous on* $[0,\, \pi]$, *and thus is bounded below by a positive quantity.*

§4 The continuity of $\mathcal{K}(x,\, y)$ on the triangle

Now we can prove the continuity of $\mathcal{K}(x,\, y)$ on the triangle $0 \le y \le x \le \pi$, which establishes the existence and compactness of the Volterra operator $\mathcal{K}$, hence the invertibility of $I + \mathcal{K}$, on each $L^2([0,\, x];\, \mathbb{R}^{d\times d})$ and $C([0,\, x];\, \mathbb{R}^{d\times d})$, $0 < x \le \pi$.

(4.1) **Theorem:** *The solution,* $\mathcal{K}(x,\, y)$, *of the transformation equation,* (2.1), *is continuous in the triangle* $0 \le y \le x \le \pi$.

Proof: Again, we work with the transposed transformation equations, (2.4):

$$\mathcal{K}(tx,\, y)^T + \mathcal{F}(y,\, tx) + \int_0^{tx} \mathcal{F}(y,\, s)\mathcal{K}(tx,\, s)^T\, ds = 0, \quad 0 \le y \le tx \le \pi. \tag{4.2}$$

We have replaced x by tx, thinking of t as a number close to 1, and we now replace y by ty, which gives

$$\mathcal{K}(tx,\, ty)^T + \mathcal{F}(ty,\, tx) + \int_0^{tx} \mathcal{F}(ty,\, s)\mathcal{K}(tx,\, s)^T\, ds = 0, \quad 0 \le y \le x \le \pi. \tag{4.3}$$

Since x is fixed, let us write $f_t(y) := \mathcal{K}(tx,\, ty)^T$, $g_t(y) := \mathcal{F}(ty,\, tx)$. We can rewrite (4.3) as

$$\mathcal{W}_{*t} f_t(y) = f_t(y) + \int_0^{x} \mathcal{F}_t(y,\, s) f_t(s)\, ds = -g_t(y), \quad 0 \le y \le x. \tag{4.4}$$

The continuity of $\mathcal{F}(y,\, s)$ and the invertibility of $\mathcal{W}_{*t}$ show that $f_t(y) = \mathcal{K}(tx,\, ty)^T$ is continuous in y on the interval $[0,\, x]$.

We next wish to show that

$$\|f_t - f_1\| = \|\mathcal{W}_{*1}^{-1} g_1 - \mathcal{W}_{*t}^{-1} g_t\| \to 0 \ \text{ as } \ t \to 1.$$

This will show the uniform convergence of $\mathcal{K}(tx,\, ty)^T$ to $\mathcal{K}(x,\, y)^T$ in $C([0,\, x];\, \mathbb{R}^{d\times d})$. Here the norm is the sup-norm on $C([0,\, x];\, \mathbb{R}^{d\times d})$. We use the continuity of $\mathcal{W}_{*t}^{-1}$ and the continuity of $\mathcal{F}(y,\, s)$ to show this, as in the proof of (3.2). The desired continuity follows.

§5 Notation: the operator $\mathcal{T} = I + \mathcal{K}$, and the transformed basis $\{\mathcal{T}\varphi_n^o\}$

Were we working with Sturm-Liouville problems, we would call the operator we will introduce here a "transmutation operator." In the present context, we will call it a "transforming operator." Let us denote by $\mathcal{T}$ the operator on $L^2([0,\, \pi];\, \mathbb{R}^d)$ defined by

$$\mathcal{T} f(x) := f(x) + \int_0^x \mathcal{K}(x,\, y) f(y)\, dy.$$

We define the *transformed basis* $\{\varphi_n(x)\}$ to be the sequence $\{\mathcal{T}\varphi_n^o(x)\}$, where n runs from 0 to ∞. We wish to study these functions in some detail. Let us set down some equivalent formulas for them:

$$\varphi_n(x) := \varphi_n^o(x) + \int_0^x \mathcal{K}(x,\, y)\varphi_n^o(y)\, dy = (I + \mathcal{K})\varphi_n^o(x) = \mathcal{T}\varphi_n^o(x). \tag{5.1}$$

We will show (among other things) that $\{\varphi_n(x)\}$ is an orthogonal and complete system, though not orthonormal (unless it coincides with $\{\varphi_n^o(x)\}$). We will prove the orthogonality after we construct the inverse of $I + \mathcal{K}$.

We can express $\mathcal{K}(x,\, y)$ in terms of the functions $\varphi_n(x)$ and $\varphi_n^o(x)$.

(5.2) **Theorem:** *For* $0 \le y \le x \le \pi$,

$$\mathcal{K}(x,\, y) = -\sum_{n=0}^{\infty} c_n\, \varphi_n(x)\, \varphi_n^o(y)^T. \tag{5.3}$$

Proof: We rewrite the transformation equation and apply the uniform convergence of the series for $\mathcal{F}(x,\, y)$:

$$\begin{aligned}
\mathcal{K}(x,\, y) &= -\,\mathcal{F}(x,\, y) - \int_0^x \mathcal{K}(x,\, t)\mathcal{F}(t,\, y)\, dt \\
&= -\sum_{n=0}^{\infty} c_n\, \varphi_n^o(x)\, \varphi_n^o(y)^T - \int_0^x \mathcal{K}(x,\, t) \sum_{n=0}^{\infty} c_n\, \varphi_n^o(t)\, \varphi_n^o(y)^T\, dt \\
&= -\sum_{n=0}^{\infty} c_n \left[\varphi_n^o(x) + \int_0^x \mathcal{K}(x,\, t)\, \varphi_n^o(t)\, dt\right] \varphi_n^o(y)^T \\
&= -\sum_{n=0}^{\infty} c_n\, \varphi_n(x)\, \varphi_n^o(y)^T,
\end{aligned} \tag{5.4}$$

as desired, by (5.1).

This expression for $\mathcal{K}(x,\, y)$ and the orthonormality of the $\varphi_n^o(x)$ give us the following result.

(5.5) **Theorem:** *For* $n = 0,\, 1,\, 2,\, \ldots,$

$$\varphi_n(0) = \varphi_n^o(0) \;\; \textit{and} \;\; \varphi_n(\pi) = \frac{\varphi_n^o(\pi)}{1 + c_n}\,. \tag{5.6}$$

Proof: The first equation in (5.6) is an immediate consequence of the definition, (5.1). To verify the second equation, we substitute the formula (5.3) into (5.1) and set $x = \pi$:

$$\varphi_n(\pi) = \varphi_n^o(\pi) - \int_0^\pi \sum_{k=0}^\infty c_k\, \varphi_k(\pi)\, \varphi_k^o(y)^T \varphi_n^o(y)\, dy, \tag{5.7}$$

We may interchange sum and integral in (5.7) , which, because of the orthonormality of the $\varphi_n^o(x)$, gives

$$\varphi_n(\pi) = \varphi_n^o(\pi) - c_n\, \varphi_n(\pi),$$

which is what we desired to show.

(5.8) **Corollary:** *For any finite non-empty subset* E *of the non-negative integers,*

$$\textit{span}\{\varphi_n(x_o) : n \in E\} = \textit{span}\{\varphi_n^o(x_o) : n \in E\}$$

when $x_o = 0$ *and when* $x_o = \pi$.

§6 The symmetry of the matrices $\mathcal{K}(x,\, x)$

Our interest in proving the symmetry of $\mathcal{K}(x,\, x)$ resides in one of the applications of $\mathcal{K}$ to Sturm-Liouville problems. The proof of symmetry also introduces, for the present paper, the study of the case when only a finite number of the coefficients are non-zero. We will use this case later as well, in the study of the inverse of the operator $I + \mathcal{K}$. We need only deal with $\mathcal{K}(x,\, x)$ for $0 < x \le \pi$, because $\mathcal{K}(0,\, 0) = -\mathcal{F}(0,\, 0)$, which is symmetric by construction.

(6.1) **Theorem:** *For each* $x \in (0,\, \pi]$, *the* $d \times d$ *matrix* $\mathcal{K}(x, x)$ *is symmetric.*

Proof: We will prove this theorem in two steps. First, we consider the "finite case," wherein only a finite number of the constants c_n are non-zero. The argument here uses continuity, the global condition $1 + c_n > 0$ and linear independence on intervals $[0,\, x]$, when $0 < x \le \pi$. The second step amounts to taking the limit as the number of non-zero terms c_n tends to infinity. The proof of (6.1) does not use the norm-continuity theorem (3.2), so it only yields uniform convergence on $[0,\, x]$ for each fixed x in $(0,\, \pi]$. We then apply (6.1) and (3.2) to deduce (in (6.18)) that the convergence is "uniform in the triangle."

The case of finitely many non-zero c_n.

Let us set $\mathcal{F}(x,\, y) = \sum_{j=1}^{N} c_j \varphi^o_{n_j k_j}(x) \varphi^o_{n_j k_j}(y)^T$, where the c_j are non-zero constants such that $1 + c_j > 0$ for $j = 1, \ldots, N$. For simplicity of notation, we abuse notation and write

$$\mathcal{F}(x,\, y) = \sum_{n=1}^{N} c_n \varphi^o_n(x) \varphi^o_n(y)^T. \tag{6.2}$$

We can write equation (6.2) as a matrix equation:

$$\mathcal{F}(x,\, y) = \Phi^o(x) C \Phi^o(y)^T, \tag{6.3}$$

where $\Phi^o(x)$ is the $d \times N$ matrix $(e_i^T \varphi^o_j(x))$, whose n-th column is $\varphi^o_n(x)$, and where $C = \operatorname{diag}(c_1, \ldots, c_N)$. Later we will need to use the relation $\Phi^o(x) = \sum_{n=1}^{N} \varphi^o_n(x) e_n^T$.

The transformation equation, which has a unique solution $\mathcal{K}(x,\, y)$, can thus be written, using (6.3), as

$$\begin{aligned}\mathcal{K}(x,\, y) &= -\,\Phi^o(x)C\Phi^o(y)^T - \int_0^x \mathcal{K}(x,\, t)\Phi^o(t)C\Phi^o(y)^T\,dt \\ &= -\left(\Phi^o(x) + \int_0^x \mathcal{K}(x,\, t)\Phi^o(t)\right)C\Phi^o(y)^T\,dt,\ 0 \le y \le x \le \pi.\end{aligned} \tag{6.4}$$

In the present case, in which only a finite number of the c_n are non-zero, the symmetry follows from the next Lemma, concerning the *form* of $\mathcal{K}(x,\, y)$ as a matrix product.

(6.5) **Lemma:** *There exist a* $d \times N$ *matrix-valued function* $\Phi(x)$ *and a symmetric* $N \times N$ *matrix-valued function* $P(x)$, *defined on* $[0, \pi]$, *such that*

$$\mathcal{K}(x,\, y) = -\Phi(x)(C + CP(y)C)\Phi(y)^T, \quad 0 \le y \le x \le \pi. \tag{6.6}$$

Proof: We define the $d \times N$ matrix-valued function $\Phi(x)$ from (6.4) by

$$\Phi(x) := \Phi^o(x) + \int_0^x \mathcal{K}(x,\, t)\Phi^o(t)\,dt, \tag{6.7}$$

(which is part of (5.1)) so that (6.4) becomes

$$\mathcal{K}(x,\, y) = -\,\Phi(x)C\Phi^o(y)^T,\ 0 \le y \le x \le \pi. \tag{6.8}$$

We will replace $\Phi^o(y)^T$ in (6.8). To do so, we construct a symmetric $N \times N$ matrix-valued function $P(x)$ such that

$$\Phi(x)\big(I + CP(x)\big) = \Phi^o(x), \quad 0 \le x \le \pi, \tag{6.9}$$

and then we will replace x by y and transpose to arrive at

$$\Phi^o(y)^T = \big(I + P(y)C\big)\Phi(y)^T.$$

When this is put into (6.8), we obtain (6.6). Thus all we need to do is prove (6.9).

First, we suppose $0 < x \le \pi$. We rewrite (6.4), using (6.8):

(6.10)

$$\mathcal{K}(x,\, y) = -\Phi(x)C\Phi^o(y)^T = -\Phi_n^o(x)C\Phi_n^o(y)^T + \int_0^x \Phi(x)C\Phi^o(t)^T\Phi_n^o(t)C\Phi_n^o(y)^T\,dt.$$

When we add $\Phi(x)C\Phi^o(y)^T + \Phi_n^o(x)C\Phi_n^o(y)^T$ to each member of these equations, and drop the first equation, (6.10) becomes

$$\Phi(x)\left(I + C\int_0^x \Phi^o(t)^T\Phi_n^o(t)\,dt\right)C\Phi^o(y)^T = \Phi_n^o(x)C\Phi_n^o(y)^T. \tag{6.11}$$

Now we define $P(x)$ by

$$P(x) := \int_0^x \Phi^o(t)^T\Phi^o(t)\,dt, \tag{6.12}$$

which is a symmetric, non-negative $N \times N$ matrix-valued function.

We can then rewrite (6.11) as

$$\Phi(x)\bigl(I + CP(x)\bigr)C\Phi^o(y)^T = \Phi^o(x)C\Phi^o(y)^T. \tag{6.13}$$

We next want to show that $\Phi(x)\bigl(I + CP(x)\bigr) = \Phi^o(x)$. Here are the details:

$$(\Phi(x)\bigl(I + CP(x)\bigr) - \Phi^o(x))C\Phi^o(y)^T = 0. \tag{6.14}$$

Since $\Phi^o(y)^T = \sum_{n=1}^N e_n\varphi_n^o(y)^T$, $(\Phi(x)\bigl(I + CP(x)\bigr) - \Phi^o(x))C\sum_{n=1}^N e_n\varphi_n^o(y)^T = 0$, so

$$\sum_{n=1}^N (\Phi(x)\bigl(I + CP(x)\bigr) - \Phi^o(x))Ce_n\varphi_n^o(y)^T = 0. \tag{6.15}$$

For each $i \in \{1, \ldots, d\}$, we thus have

$$\sum_{n=1}^N \Bigl[e_i^T(\Phi(x)\bigl(I + CP(x)\bigr) - \Phi^o(x))Ce_n\Bigr]\varphi_n^o(y)^T = 0, \quad \text{for } \; y \in [0,\, x]. \tag{6.16}$$

The set $\{\varphi_1^o(y)^T, \ldots, \varphi_N^o(y)^T\}$ of vector functions is linearly independent on $[0, x]$. Therefore $\Big[\Phi(x)\big(I + CP(x)\big) - \Phi^o(x)\Big]C = 0$. Now C is invertible, and therefore we see that

$\Phi(x)\big(I + CP(x)\big) = \Phi^o(x)$, as desired. Equation (6.9) holds at $x = 0$ as well, by continuity.

Having now shown that the kernels $\mathcal{K}(x, x)$ induced by a "finite case" $\mathcal{F}(x, y)$ are symmetric, we turn to our second step, the limit as the number of non-zero terms tends to infinity. We will return to the original notation, in which n runs from 0 to ∞.

(6.17) **Lemma:** *For each* $x \in (0, \pi]$, $\mathcal{K}_N(x, y) \to \mathcal{K}(x, y)$ *uniformly for* $0 \le y \le x$, *as* $N \to \infty$.

Proof: We fix x in $(0, \pi]$, and, for $0 \le y \le x$ we let $Tf = T_x f$ denote the $d \times d$ matrix-valued function

$$Tf(y) = \int_0^x \mathcal{F}(y, t) f(t)\, dt,$$

that we have called $\mathcal{F}_x f(y)$ before, where

$$\mathcal{F}(x, y) = \sum_{n=0}^{\infty} c_n\, \varphi_n^o(x)\, \varphi_n^o(y)^T \quad \text{for} \quad 0 \le x \le \pi \quad \text{and} \quad 0 \le y \le \pi,$$

and where $f(t)$ is a continuous $d \times d$ matrix-valued function. The real numbers c_n, $n = 0, 1, 2, \ldots$ were chosen so that $1 + c_n > 0$ for all n. This ensures, *via* the transformation equation, that $I + T$ is invertible on $L^2([0, x], \mathbb{R}^{d \times d})$, and on $C([0, x], \mathbb{R}^{d \times d})$ as well. We will use the absolute convergence of the series $\sum c_n$ and the uniform bound A on the functions $|\varphi_n^o(x)|$ for $0 \le x \le \pi$.

In the transformation equation,

$$\mathcal{K}(x,\, y) + \mathcal{F}(x,\, y) + \int_0^x \mathcal{K}(x,\, t)\mathcal{F}(t, y)\, dt = 0, \quad 0 \le y \le x \le \pi,$$

we transpose and write $\tilde{\mathcal{K}}(y) := \mathcal{K}(x,\, y)^T$, and likewise abbreviate other functions of two variables that are to be treated as functions of the single variable y because we have fixed x. Thus the transposed version of the transformation equation can be written

$$\tilde{\mathcal{K}}(y) + \tilde{\mathcal{F}}(y) + \int_0^x \mathcal{F}(y, t)\tilde{\mathcal{K}}(t)\, dt = 0, \quad 0 \le y \le x.$$

We can write the last equation as

$$\tilde{\mathcal{K}} + \tilde{\mathcal{F}} + T\tilde{\mathcal{K}} = 0.$$

We have similar operators T_N, functions $\tilde{\mathcal{F}}_N$, as well as the functions $\tilde{\mathcal{K}}_N$, that are the solutions of equations

$$\tilde{\mathcal{K}}_N + \tilde{\mathcal{F}}_N + T_N\tilde{\mathcal{K}}_N = 0.$$

These were obtained by setting to zero all the c_k with $k > N$. We note that, if some of the c_n are zero, we ignore the corresponding basis functions, and the corresponding terms in the series that we use. In particular, we are only interested in N when N is so large that there are some non-zero c_n for $n < N$.

We intend to show that $\tilde{\mathcal{K}}_N(y) \to \tilde{\mathcal{K}}(y)$ uniformly in $0 \le y \le x$, as $N \to \infty$. Since we have shown that $\tilde{\mathcal{K}}_N(x)$ is symmetric, it will follow that $\mathcal{K}(x, x)^T = \tilde{\mathcal{K}}(x)$ is symmetric.

We begin by inserting T into the equations

$$\tilde{\mathcal{K}}_N + \tilde{\mathcal{F}}_N + T_N\tilde{\mathcal{K}}_N = 0:$$

$$(I+T_N)\tilde{\mathcal{K}}_N = -\tilde{\mathcal{F}}_N = (I+T)\tilde{\mathcal{K}}_N + (T_N - T)\tilde{\mathcal{K}}_N.$$

We can now insert the "limit" transformation equation:

$$\tilde{\mathcal{F}} - \tilde{\mathcal{F}}_N = (I+T)(\tilde{\mathcal{K}}_N - \tilde{\mathcal{K}}) + (T_N - T)\tilde{\mathcal{K}}_N.$$

Let us write $\tilde{\mathcal{R}}_N := \tilde{\mathcal{F}} - \tilde{\mathcal{F}}_N$, and then rewrite the last equation:

$$(I+T)(\tilde{\mathcal{K}}_N - \tilde{\mathcal{K}}) = \tilde{\mathcal{R}}_N - (T_N - T)\tilde{\mathcal{K}}_N,$$

or as

$$(I+T)(\tilde{\mathcal{K}} - \tilde{\mathcal{K}}_N) = (T - T_N)\tilde{\mathcal{K}}_N - \tilde{\mathcal{R}}_N.$$

We observe that $\tilde{\mathcal{R}}_N(y) = \sum_{n=N+1}^{\infty} c_n\, \varphi_n^o(x)\, \varphi_n^o(y)^T$ converges uniformly to zero because of the absolute convergence of $\sum c_k$. We wish to show that $(T - T_N)\tilde{\mathcal{K}}_N$ converges uniformly to zero also. Then the invertibility of $I+T$ on $C([0,x],\mathbb{R}^{d\times d})$ gives us the desired uniform convergence to zero of $\tilde{\mathcal{K}} - \tilde{\mathcal{K}}_N$.

To show that $(T-T_N)\tilde{\mathcal{K}}_N$ converges uniformly to zero we will use the Schwarz inequality, estimates on the kernel of the operator $T - T_N$, and a bound on the L^2 norm of $\tilde{\mathcal{K}}_N$, obtained using these facts: the operator norm of $T - T_N$ on $L^2([0,x],\mathbb{R}^{d\times d})$ tends to zero with N, and, the L^2 norms of the $\tilde{\mathcal{F}}_N$ are bounded.

To show that the L^2 norms of the $\tilde{\mathcal{F}}_N$ are bounded, we estimate crudely:

$$\int_0^x \left|\tilde{\mathcal{F}}_N(y)\right|^2 dy = \int_0^x |\mathcal{F}_N(x,\, y)|^2\, dy = \int_0^x \left|\sum_{n=0}^{N} c_n\, \varphi_n^o(x)\, \varphi_n^o(y)^T\right|^2 dy$$
$$\le \int_0^x \left|\sum_{n=0}^{N} |c_n| A^2\right|^2 dy \le \pi A^4 \left(\sum_{n=0}^{\infty} |c_n|\right)^2.$$

We now know that

$$\left\|\tilde{\mathcal{F}}_N\right\|_2 < 2A^2 \sum_{n=0}^{\infty} |c_n|.$$

To show that the operator norm of $T - T_N$ on $L^2([0,x], \mathbb{R}^{d\times d})$ tends to zero with N, we view the operator $T - T_N = \sum_{k=N+1}^{\infty} U_k$ as a sum of the rank-one operators $U_k f(y) := c_k \langle f,\, \varphi_k^o\rangle \varphi_k^o(y),\ 0 \le y \le x$. We then have

$$\|(T - T_N)f\|_2 \le \sum_{k>N} |c_k|\, \|f\|_2 \to 0 \ \text{ as } \ N \to \infty.$$

Let $\alpha := 1/\|(I+T)^{-1}\|_{2,2}$, the reciprocal of the norm of $(I+T)^{-1}$ as an operator on $L^2([0,x], \mathbb{R}^{d\times d})$. Let N_α be so large that $N > N_\alpha$ implies that $\|T - T_N\|_{2,2} < \alpha/2$. Then, from the equation

$$-\tilde{\mathcal{F}}_N = (I+T)\tilde{\mathcal{K}}_N + (T_N - T)\tilde{\mathcal{K}}_N$$

and our estimate for $\left\|\tilde{\mathcal{F}}_N\right\|_2$ we deduce that, if $N > N_\alpha$, then

$$2A^2 \sum_{n=0}^{\infty} |c_n| \ge \left\|\tilde{\mathcal{F}}_N(y)\right\|_2 \ge \|(I+T)\tilde{\mathcal{K}}_N\|_2 - \|T_N - T\|_{2,2}\|\tilde{\mathcal{K}}_N\|_2 \ge \alpha\|\tilde{\mathcal{K}}_N\|_2 - (\alpha/2)\|\tilde{\mathcal{K}}_N\|_2.$$

That is,

$$\left\|\tilde{\mathcal{K}}_N\right\|_2 \le (4A^2/\alpha) \sum_{n=0}^{\infty} |c_n|$$

for $N > N_\alpha$. Now we can complete the proof.

$$(T_N - T)\tilde{\mathcal{K}}_N(y) = \int_0^x \sum_{k>N}^{\infty} c_k \varphi_k^o(y)\varphi_k^o(t)^T \tilde{\mathcal{K}}_N(t)\, dt,$$

so we have

$$\begin{aligned}\left|(T_N - T)\tilde{\mathcal{K}}_N(y)\right| &\le \left\|\tilde{\mathcal{K}}_N\right\|_2 \left\{ \int_0^x \left| \sum_{k>N}^{\infty} c_k \varphi_k^o(y)\varphi_k^o(t)^T \right|^2 dt \right\}^{1/2} \\ &\le (4A^2/\alpha) \sum_{n=N+1}^{\infty} |c_n| \sqrt{x} \left(2A^2 \sum_{k>n}^{\infty} |c_n| \right)^{1/2},\end{aligned}$$

and this converges to zero uniformly in y on $[0,\, x]$.

Since x was arbitrary, $\mathcal{K}(x,\, x)$ is symmetric for all x in $(0,\, \pi]$, and the proof is done.

We can combine the proof of this lemma and the lower bound on the operators $(I+T)^{-1}$ that we obtained in Theorem (3.2) to arrive at the following theorem.

(6.18) **Theorem:** *On the triangle* $0 \le y \le x \le \pi$, $\mathcal{K}_N(x,\, y) \to \mathcal{K}(x,\, y)$ *uniformly as* $N \to \infty$.

§7 A matrix formula for $\mathcal{K}(x,\, y)$, in the "finite case"

We return to the case when $\mathcal{F}$ is defined with N non-zero c_n. Our formula will be used in the construction of the inverse of $I+\mathcal{K}$. In the general case, the formula for the inverse will be proved by taking a limit, using (6.18).

Let us recall equations (6.8) and (6.9), namely

$$\mathcal{K}(x,\, y) = -\Phi(x) C \Phi^o(y)^T, \quad 0 \le y \le x \le \pi, \tag{7.1}$$

and

$$\Phi(x)\big(I + CP(x)\big) = \Phi^o(x), \quad 0 \le x \le \pi. \tag{7.2}$$

These equations suggest our formula, an alternative to (6.6):

(7.3) **Theorem:** If $\mathcal{F}$ is defined with N non-zero c_n, then

$$\mathcal{K}(x,\, y) = -\Phi^o(x)\big(I + CP(x)\big)^{-1} C \Phi^o(y)^T, \quad 0 \le y \le x \le \pi. \tag{7.4}$$

Remark: We will find it convenient later to write (7.4) as

$$\mathcal{K}(x,\, y) = -\Phi^o(x) C \big(C + CP(x)C\big)^{-1} C \Phi^o(y)^T, \quad 0 \le y \le x \le \pi. \tag{7.5}$$

Proof: The following lemma is what we need to verify (7.4).

(7.6) **Lemma:** *For each x in $[0,\,\pi]$, $I + CP(x)$ is invertible.*

Proof: Since $P(0) = 0$, and $P(\pi) = I$, $I + CP(x)$ is invertible at the endpoints (we assumed that $1 + c_n > 0$ for all n). Suppose that $I + CP(x)$ is *not* invertible for some x, $0 < x < \pi$. Then there exists a non-zero column vector $\eta(x)$, of size $N \times 1$, such that $\eta(x)^T(I + CP(x)) = 0$. We know, by the transformation equation, *via* (6.9), that the equation

$$\Phi(x)(I + CP(x)) = \Phi^o(x)$$

has a solution, $\Phi(x)$, so we have

$$(\Phi(x) + v\eta(x)^T)(I + CP(x)) = \Phi^o(x), \tag{7.7}$$

where v is an arbitrary non-zero d-dimensional vector. Let us consider the function $Z(x,\,y)$, defined by

$$Z(x,\,y) := -(\Phi(x) + v\eta(x)^T)C\Phi^o(y)^T, \quad 0 \le y \le x. \tag{7.8}$$

Let us show that $Z(x,\,y) = \mathcal{K}(x,\,y)$ for $0 \le y \le x$. When we put $Z(x,\,y)$ into the left-hand side of the transformation equation in place of $\mathcal{K}(x,\,y)$, we get

$$\begin{aligned} Z(x,\,y) + \mathcal{F}(x,\,y) + &\int_0^x Z(x,\,t)\mathcal{F}(t,y)\,dt \\ &= -\,(\Phi(x) + v\eta(x)^T)C\Phi^o(y)^T + \Phi^o(x)C\Phi^o(y)^T \\ &\quad -\,(\Phi(x) + v\eta(x)^T)C\int_0^x \Phi^o(t)^T\Phi^o(t)\,dt\,C\Phi^o(y)^T. \end{aligned} \tag{7.9}$$

We recall that $\int_0^x \Phi^o(t)^T\Phi^o(t)\,dt = P(x)$, the matrix defined in (6.12). Then, from (7.9) and (7.7)

$$
\begin{aligned}
Z(x,\,y)+\mathcal{F}(x,\,y)+&\int_0^x Z(x,\,t)\mathcal{F}(t,y)\,dt\\
&=-\,(\Phi(x)+v\eta(x)^T)C\Phi^o(y)^T+\Phi^o(x)C\Phi^o(y)^T\\
&\quad-(\Phi(x)+v\eta(x)^T)C\,P(x)\,C\Phi^o(y)^T\\
&=-\,(\Phi(x)+v\eta(x)^T)(I+CP(x))C\Phi^o(y)^T+\Phi^o(x)C\Phi^o(y)^T\\
&=-\,\Phi^o(x)C\Phi^o(y)^T+\Phi^o(x)C\Phi^o(y)^T=0,\quad\text{for }\ 0\le y\le x.
\end{aligned}
$$

Therefore $Z(x,\,y)$ is a solution of the transformation equation, at least for this x and for $0 \le y \le x$. By uniqueness, *which holds for each* $x \in (0,\,\pi]$, $Z(x,\,y) = \mathcal{K}(x,\,y)$ on $[0, x]$. That is, for $0 \le y \le x$,

$$Z(x,\,y)=-(\Phi(x)+v\eta(x)^T)C\,\Phi^o(y)^T=\mathcal{K}(x,\,y)=-\Phi(x)\,C\Phi^o(y)^T,$$

$$\text{so that }\ v\eta(x)^TC\,\Phi^o(y)^T\equiv 0$$

for $y \in [0, x]$. This can be rewritten as the rank-one matrix equation

$$v\sum_{n=1}^N c_n\eta_n(x)\varphi_n^o(y)^T=0,\quad 0\le y\le x.$$

Since $v \neq 0$, we can multiply on the left by v^T and conclude that

$$\sum_{n=1}^N c_n\eta_n(x)\varphi_n^o(y)^T=0\ \text{ for }\ 0\le y\le x.$$

By the linear independence of the set $\{\varphi_n^o(y)^T\}$ on $[0,\,x]$, we must have $c_n\eta_n(x) = 0$ for each n. We have assumed that all the $c_n \neq 0$, so $\eta(x) = 0$, and this gives a contradiction. Thus, $I + CP(x)$ is invertible, for all $x \in (0,\,\pi]$.

§8 A “closed-form” formula for the inverse of $I + \mathcal{K}$.

Instead of using the Neumann series to find the inverse of $I + \mathcal{K}$, we can express the inverse directly, using a formula that will turn out to be part of a natural extension of the transformation equation to the whole square $0 \le x \le \pi, \quad 0 \le y \le \pi$.

(8.1) **Theorem:** The operator $I + \mathcal{K}$ is invertible, and its inverse has the form $I + \mathcal{L}$, where $\mathcal{L}$ is a Volterra integral operator, with kernel

$$\mathcal{L}(x, y) = \mathcal{F}(x, y) + \int_0^y \mathcal{F}(x, u)\mathcal{K}(y, u)^T du, \quad \text{for} \quad 0 \le y \le x \le \pi. \tag{8.2}$$

We prove this first in the "finite case" – when only finitely many of the constants c_n used to define $\mathcal{F}(x, y)$ are non-zero. In the finite case, we can express $\mathcal{K}(x, y)$ and $\mathcal{L}(x, y)$ as products of matrices, and then show, by computing, that the equations

$$\mathcal{K}(x, y) + \mathcal{L}(x, y) + \int_y^x \mathcal{K}(x, t)\mathcal{L}(t, y)\, dt = 0, \quad 0 \le y \le x \le \pi \tag{8.3}$$

and

$$\mathcal{K}(x, y) + \mathcal{L}(x, y) + \int_y^x \mathcal{L}(x, t)\mathcal{K}(t, y)\, dt = 0, \quad 0 \le y \le x \le \pi \tag{8.4}$$

are true, which implies that $I + \mathcal{L} = (I + \mathcal{K})^{-1}$. We will then take limits to prove the same result in the general case.

In the "finite case," we have (7.5), namely

$$\mathcal{K}(x, y) = -\Phi^o(x)C\big(C + CP(x)C\big)^{-1}C\Phi^o(y)^T =: -\Phi^o(x)CD(x)C\Phi^o(y)^T, \tag{8.5}$$

in which we define, for brevity, $D(x) := \big(C + CP(x)C\big)^{-1}$. For later use, we denote the inverse of $D(x)$ by $N(x) = C + CP(x)C$.

We convert (8.2) into a matrix product similar to (8.5):

$$\mathcal{L}(x,\,y) = \Phi^o(x)C\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T = \Phi^o(x)CD(y)C\Phi^o(y)^T. \tag{8.6}$$

Here are the details of the computation showing (8.6):

$$\begin{aligned}
\mathcal{L}(x,\,y) =&\mathcal{F}(x,\,y) + \int_0^y \mathcal{F}(x,\,u)K(y,\,u)^T du\\
=&\Phi^o(x)C\Phi^o(y)^T\\
&- \int_0^y \Phi^o(x)C\Phi^o(u)^T\Phi^o(u)C\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T du\\
=&\Phi^o(x)C\Phi^o(y)^T\\
&- \Phi^o(x)CP(y)C\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T\\
=&\Phi^o(x)C\Phi^o(y)^T\\
&- \Phi^o(x)(-C + C + CP(y)C)\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T\\
=&\Phi^o(x)C\Phi^o(y)^T\\
&+ \Phi^o(x)C\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T - \Phi^o(x)C\Phi^o(y)^T\\
=&\Phi^o(x)C\big(C + CP(y)C\big)^{-1}C\Phi^o(y)^T = \Phi^o(x)CD(y)C\Phi^o(y)^T.
\end{aligned}$$

Thus we have, on collecting the expressions for $\mathcal{K}(x,\,y)$ and $\mathcal{L}(x,\,y)$,

$$\mathcal{K}(x,\,y) + \mathcal{L}(x,\,y) = -\Phi^o(x)C\big[D(x) - D(y)\big]C\Phi^o(y)^T. \tag{8.7}$$

We seek a similar expression for the kernel giving the composite operator $\mathcal{K}\mathcal{L}$:

$$\begin{aligned}
\int_y^x \mathcal{K}(x,\,t)\mathcal{L}(t,\,y)\,dt =& -\int_y^x \Phi^o(x)CD(x)C\Phi^o(t)^T\Phi^o(t)CD(y)C\Phi^o(y)^T dt\\
=& -\Phi^o(x)CD(x)C[P(x) - P(y)]CD(y)C\Phi^o(y)^T.
\end{aligned} \tag{8.8}$$

We can rewrite (8.7) as

$$
\begin{aligned}
\mathcal{K}(x,\, y) + \mathcal{L}(x,\, y) &= -\,\Phi^o(x) C \big[D(x) - D(y) \big] C \Phi^o(y)^T \\
&= -\,\Phi^o(x) C \big[D(x) N(y) D(y) - D(x) N(x) D(y) \big] C \Phi^o(y)^T \\
&= -\,\Phi^o(x) C D(x) \big[N(y) - N(x) \big] D(y) C \Phi^o(y)^T \\
&= -\,\Phi^o(x) C D(x) \big[C P(y) C - C P(x) C \big] D(y) C \Phi^o(y)^T \\
&= -\int_y^x \mathcal{K}(x,\, t) \mathcal{L}(t,\, y)\, dt,
\end{aligned}
\tag{8.9}
$$

using (8.8), and this is one of the equations desired (i.e., (8.3)) .

It remains to show that $\mathcal{K} + \mathcal{L} + \mathcal{L}\mathcal{K} = 0$. We will show this by showing that the corresponding Volterra kernels satisfy the equation pointwise, for $0 \le y \le x \le \pi$. We have

$$
\begin{aligned}
\int_y^x \mathcal{L}(x,\, t) \mathcal{K}(t,\, y)\, dt &= -\int_y^x \Phi^o(x) C D(t) C \Phi^o(t)^T \Phi^o(t) C D(t) C \Phi^o(y)^T dt \\
&= -\,\Phi^o(x) C \int_y^x D(t) C \Phi^o(t)^T \Phi^o(t) C D(t)\, dt\, C \Phi^o(y)^T.
\end{aligned}
\tag{8.10}
$$

We can show that the integrand, $D(t) C \Phi^o(t)^T \Phi^o(t) C D(t)$, is the negative of the derivative of $D(t)$. To do so, we introduce the identity matrices $N(t)D(t)$ and $N(s)D(s)$ into the difference quotient, and again write the factors $N(\cdot)$ as $C + CP(\cdot)C$:

$$
\begin{aligned}
\frac{D(s) - D(t)}{s - t} &= \frac{D(s) N(t) D(t) - D(s) N(s) D(t)}{s - t} = \frac{D(s) \big[N(t) - N(s) \big] D(t)}{s - t} \\
&= \frac{D(s) \big[C + C P(t) C - C - C P(s) C \big] D(t)}{s - t} \\
&= \frac{D(s) C \big[P(t) - P(s) \big] C D(t)}{s - t} \\
&\quad \to -D(t) C P'(t) C D(t) = -D(t) C \Phi^o(t)^T \Phi^o(t) C D(t).
\end{aligned}
\tag{8.11}
$$

Hence

$$
\begin{aligned}
\int_y^x \mathcal{L}(x,\,t)\mathcal{K}(t,\,y)\,dt &= -\,\Phi^o(x)C\int_y^x D(t)C\Phi^o(t)^T\Phi^o(t)CD(t)\,dt\,C\Phi^o(y)^T\\
&= -\,\Phi^o(x)C\int_y^x \big(-D'(t)\big)\,dt\,C\Phi^o(y)^T\\
&= \Phi^o(x)C\big[D(x)-D(y)\big]C\Phi^o(y)^T\\
&= -\,\big(\mathcal{K}(x,\,y)+\mathcal{L}(x,\,y)\big),
\end{aligned}
\tag{8.12}
$$

by (8.7). This completes the proof of the Theorem, in the "finite case."

We rewrite (8.2), (8.3) and (8.4), inserting the parameter N:

$$\mathcal{L}_N(x,\,y)=\mathcal{F}_N(x,\,y)+\int_0^y \mathcal{F}_N(x,\,u)\mathcal{K}_N(y,\,u)^T du,\quad \text{for}\ \ 0\le y\le x\le\pi; \tag{8.13}$$

$$\mathcal{K}_N(x,\,y)+\mathcal{L}_N(x,\,y)+\int_y^x \mathcal{K}_N(x,\,t)\mathcal{L}_N(t,\,y)\,dt=0,\quad 0\le y\le x\le\pi; \tag{8.14}$$

$$\mathcal{K}_N(x,\,y)+\mathcal{L}_N(x,\,y)+\int_y^x \mathcal{L}_N(x,\,t)\mathcal{K}_N(t,\,y)\,dt=0,\quad 0\le y\le x\le\pi. \tag{8.15}$$

Now $\mathcal{K}_N(x,\,y)$ converges uniformly to $\mathcal{K}(x,\,y)$ in the triangle $0\le y\le x\le\pi$ (see (6.18)). By (8.13), $\mathcal{L}_N(x,\,y)$ converges uniformly in the same triangle, to $\mathcal{L}(x,\,y)$. Thus the equations (8.14) and (8.15) converge pointwise to (8.3) and (8.4), respectively. This completes the proof.

§9 The extended transformation equation

It is an interesting curiosity that the transformation equation "carries the seeds of its own inversion." In other words, the transformation equation itself can be used to find the kernel $\mathcal{L}$ that appears in the inverse, $I+\mathcal{L}$, of $I+\mathcal{K}$. Let us make an observation that we have alluded to before.

The right-hand side of the following form of the transformation equation,

$$(9.1)\qquad -\mathcal{K}(x,\, y) = \mathcal{F}(x,\, y) + \int_0^x \mathcal{K}(x,\, t)\mathcal{F}(t,\, y)\, dt, \quad 0 \le y \le x \le \pi,$$

makes sense for all $(x,\, y)$ *in the square* $[0,\, \pi] \times [0,\, \pi]$.

Therefore, let us extend $\mathcal{K}(x,\, y)$ to the kernel $\mathcal{K}_\#(x,\, y)$ defined on the whole square, $0 \le x \le \pi, \;\; 0 \le y \le \pi,$ by

$$(9.2)\qquad \mathcal{K}_\#(x,\, y) := -\mathcal{F}(x,\, y) - \int_0^x \mathcal{K}(x,\, t)\mathcal{F}(t,\, y)\, dt.$$

Then

$$\mathcal{K}_\#(x,\, y) = \mathcal{K}(x,\, y), \quad 0 \le y \le x \le \pi,$$

by definition, while, for $0 < x < y < \pi,$ or for $(x,\, y)$ on the boundary of the square above the main diagonal, we see, from (8.2), namely

$$(9.2.5)\qquad \mathcal{L}(y,\, x) = \mathcal{F}(y,\, x) + \int_0^x \mathcal{F}(y,\, u)\mathcal{K}(x,\, u)^T du, \quad \text{for} \;\; 0 \le x \le y \le \pi,$$

but with x and y interchanged here, that

$$(9.3)\qquad \mathcal{K}_\#(x,\, y) = -\mathcal{L}(y,\, x)^T$$

for $0 < x < y < \pi,$ or for $(x,\, y)$ on the boundary of the square above the main diagonal. This holds for $0 \le x \le y \le \pi,$ by continuity. We note that, if we define, in the same spirit,

$$(9.4)\qquad \mathcal{L}_\#(x, y) := \mathcal{F}(x, y) + \int_0^y \mathcal{F}(x, t)K(y, t)^T\, dt,$$

for $0 \le x \le \pi$ and $0 \le y \le \pi,$ then we have $\mathcal{L}_\#(x, y) = -\mathcal{K}_\#(y, x)^T.$

Let us return to (9.2.5), transpose both sides, and apply the same argument that was used to prove Theorem (5.2). The result is that

$$\begin{aligned}\mathcal{L}(y,\,x)^T &= \mathcal{F}(x,\,y) + \int_0^x \mathcal{K}(x,\,u)\mathcal{F}(u,\,y)\,du,\\ &= \sum_{n=0}^{\infty} c_n\,\varphi_n(x)\,\varphi_n^o(y)^T = -\mathcal{K}_\#(x,\,y) \quad \text{for} \quad 0 \le x \le y \le \pi.\end{aligned} \tag{9.5}$$

Let us express the content of this section as a theorem.

(9.6) **Theorem:** *The functions* $\mathcal{K}_\#$ *and* $\mathcal{L}_\#$, *defined in* (9.2) *and* (9.4), *are given by the following series for all* $0 \le x \le \pi$ *and* $0 \le y \le \pi$:

$$\mathcal{K}_\#(x,\,y) = -\sum_{n=0}^{\infty} c_n\,\varphi_n(x)\,\varphi_n^o(y)^T, \tag{9.7}$$

and

$$\mathcal{L}_\#(x,\,y) = \sum_{n=0}^{\infty} c_n\,\varphi_n^o(x)\,\varphi_n(y)^T. \tag{9.8}$$

These functions will be used as kernels for integral operators.

§10 The orthogonality of the transformed o.n. basis

We recall (5.1), in which we defined the transformed functions $\varphi_n(x)$:

$$\varphi_n(x) := \varphi_n^o(x) + \int_0^x \mathcal{K}(x,\,y)\varphi_n^o(y)\,dy. \tag{10.1}$$

Here, we will demonstrate the orthogonality of the transformed system $\{\varphi_n(x)\}$, and find formulas for (the squares of) its "norming constants," $\alpha_n := \|\varphi_n\|$.

(10.2) **Theorem:** *For all non-negative integers* m *and* n,

$$\langle\varphi_m,\,\varphi_n\rangle := \int_0^\pi \varphi_m(x)^T\varphi_m(x)\,dx = \frac{\delta_{mn}}{1+c_n}, \quad \textit{and} \quad \alpha_n^2 := \|\varphi_n\|^2 = \frac{1}{1+c_n}\,. \tag{10.3}$$

Proof: We use $\mathcal{L}_{\#}(x,y,)$ defined in (9.4), and several identities involving related integral operators.

Much of the argument can be expressed in terms of operators. For reference we list the definitions, for a function $g \in L^2([0,\,\pi];\,\mathbb{R}^d)$, of the following operators, acting on g:

$$\begin{aligned} \mathcal{K}g(x) &:= \int_0^x \mathcal{K}(x,\,y)g(y)\,dy, \quad \mathcal{L}g(x) := \int_0^x \mathcal{L}(x,\,y)g(y)\,dy, \\ \mathcal{F}g(x) &:= \int_0^\pi \mathcal{F}(x,\,y)g(y)\,dy, \quad \mathcal{L}_{\#}g(x) := \int_0^\pi \mathcal{L}_{\#}(x,\,y)g(y)\,dy. \end{aligned} \tag{10.3.5}$$

Of these, only the first two are Volterra operators. We need their dual operators:

$$\mathcal{K}^*g(x) = \int_x^\pi \mathcal{K}(y,\,x)^T g(y)\,dy, \quad \mathcal{L}^*g(x) = \int_x^\pi \mathcal{L}(y,\,x)^T g(y)\,dy. \tag{10.3.7}$$

We need to recall that $\mathcal{F}$ is self-adjoint:

$$\mathcal{F}^*g(x) = \int_0^\pi \mathcal{F}(y,\,x)^T g(y)\,dy = \int_0^\pi \mathcal{F}(x,\,y)g(y)\,dy = \mathcal{F}g(x).$$

We also need the kernel for the product operator $\mathcal{F}\mathcal{K}^*$:

$$\begin{aligned} \mathcal{F}\mathcal{K}^*g(x) &= \int_0^\pi \mathcal{F}(x,\,t)\mathcal{K}^*g(t)\,dt = \int_0^\pi \mathcal{F}(x,\,t)\int_t^\pi \mathcal{K}(y,\,t)^T g(y)\,dy\,dt \\ &= \int_0^\pi \left(\int_0^y \mathcal{F}(x,\,t)\mathcal{K}(y,\,t)^T dt\right) g(y)\,dy. \end{aligned} \tag{10.4}$$

It follows from (10.4) and (9.4), the definition of $\mathcal{L}_{\#}(x,\,y)$, that in terms of the operators we have defined,

$$\mathcal{L}_{\#} = \mathcal{F} + \mathcal{F}\mathcal{K}^* = \mathcal{F}(I + \mathcal{K}^*). \tag{10.5}$$

We may also use (9.2), (9.3), (9.4) and the formulas in (10.3.5) and (10.3.7) to calculate $\mathcal{L}_{\#}g$:

$$\begin{aligned} \mathcal{L}_{\#}g(x) &= \int_0^\pi \mathcal{L}_{\#}(x,\,y)g(y)\,dy = \int_0^x \mathcal{L}_{\#}(x,\,y)g(y)\,dy + \int_x^\pi \mathcal{L}_{\#}(x,\,y)g(y)\,dy \\ &= \int_0^x \mathcal{L}(x,\,y)g(y)\,dy - \int_x^\pi \mathcal{K}(y,\,x)^T g(y)\,dy \\ &= \mathcal{L}g(x) - \mathcal{K}^*g(x). \end{aligned} \tag{10.6}$$

Thus by (10.5) and (10.6),

$$\mathcal{L}_{\#} = \mathcal{F}(I + \mathcal{K}^*) = \mathcal{L} - \mathcal{K}^*.$$

By eliminating $\mathcal{L}_{\#}$, then adding I to both sides, we can write

$$I + \mathcal{L} = I + \mathcal{K}^* + \mathcal{F}(I + \mathcal{K}^*) = (I + \mathcal{F})(I + \mathcal{K}^*),$$

or instead, the equation we really need,

$$I + \mathcal{L}^* = (I + \mathcal{K})(I + \mathcal{F}^*) = (I + \mathcal{K})(I + \mathcal{F}). \tag{10.7}$$

Since $(I + \mathcal{L})(I + \mathcal{K}) = I$, we can write, for all non-negative integers m and n,

$$\delta_{mn} = \langle \varphi_m^o, \varphi_n^o \rangle = \langle \varphi_m^o, (I + \mathcal{L})(I + \mathcal{K})\varphi_n^o \rangle = \langle (I + \mathcal{L}^*)\varphi_m^o, (I + \mathcal{K})\varphi_n^o \rangle. \tag{10.8}$$

From (10.7)

$$(I + \mathcal{L}^*)\varphi_m^o = (I + \mathcal{K})(I + \mathcal{F})\varphi_m^o = (I + \mathcal{K})(1 + c_m)\varphi_m^o, \tag{10.9}$$

because $\mathcal{F}\varphi_m^o(x) = \int_0^\pi \sum_{n=0}^\infty c_n \, \varphi_n^o(x) \, \varphi_n^o(y)^T \varphi_m^o(y) \, dy = c_m \varphi_m^o(x)$. Thus (10.8) becomes, by definition of the φ_n,

(10.10)

$$\delta_{mn} = \langle (I + \mathcal{L}^*)\varphi_m^o, (I + \mathcal{K})\varphi_n^o \rangle = \langle (I + \mathcal{K})(1 + c_m)\varphi_m^o, (I + \mathcal{K})\varphi_n^o \rangle = (1 + c_m)\langle \varphi_m, \varphi_n \rangle,$$

which is what we had to show.

the equation (10.7) gives us a formula for recovering $\mathcal{F}$ from $\mathcal{K}$ and $\mathcal{L}$.

(10.10.5) **Corollary:**

$$\mathcal{F} = (I + \mathcal{L})(\mathcal{L}^* - \mathcal{K}). \tag{10.10.7}$$

Proof: From (10.7) we have $I+\mathcal{L}^* = (I+\mathcal{K})(I+\mathcal{F})$. Multiplying on the left by $I+\mathcal{L}$ gives $\mathcal{F} = (I+\mathcal{L})(I+\mathcal{L}^*)-I = \mathcal{L}+\mathcal{L}^*+\mathcal{L}\mathcal{L}^* = -\mathcal{K}-\mathcal{L}\mathcal{K}+\mathcal{L}^*+\mathcal{L}\mathcal{L}^* = -\mathcal{K}+\mathcal{L}^*-\mathcal{L}(\mathcal{K}-\mathcal{L}^*)$, as desired. We used the identity $\mathcal{L}+\mathcal{K}+\mathcal{L}\mathcal{K}=0$, which holds because $I+\mathcal{L}$ and $I+\mathcal{K}$ are inverses of each other.

As another corollary, we conclude that $\alpha_n^2 := \int_0^\pi \varphi_n(x)^T\varphi_n(x)\,dx = 1/(1+c_n)$. Further, since $\varphi_n = (I+\mathcal{K})\varphi_n^o$, and $I+\mathcal{K}$ is invertible, the system $\{\sqrt{1+c_n}\,\varphi_n\}$ is orthonormal and complete. This gives the following Parseval equation for the new system:

(10.11) **Theorem:** *For all* $f \in L^2([0,\,\pi];\,\mathbb{R}^d)$,

$$\|f\|^2 = \sum_{n=0}^{\infty}(I+c_n)\mathcal{F}_n^2, \quad \textit{where} \qquad \mathcal{F}_n := \int_0^\pi \varphi_n(x)^T f(x)\,dx.$$

§11 The transformation equation for the inverse kernel

(11.1) **Theorem:** *Let*

$$\mathcal{G}(x,\,y) := -\sum_{n=0}^{\infty} c_n\,\varphi_n(x)\,\varphi_n(y)^T. \tag{11.2}$$

Then

$$\mathcal{L}(x,\,y)+\mathcal{G}(x,\,y)+\int_0^x \mathcal{L}(x,\,t)\mathcal{G}(t,\,y)\,dt = 0 \quad \textit{for} \quad 0 \le y \le x \le \pi.$$

Proof: We recall, from (5.2), that

$$\mathcal{K}(x,\,y) = -\sum_{n=0}^{\infty} c_n\varphi_n(x)\varphi_n^o(y)^T \quad \text{for} \quad 0 \le y \le x \le \pi.$$

Then, by (8.1), and the definition of $\mathcal{F}(x, y)$,

$$\begin{aligned}\mathcal{L}(x,y) =&\mathcal{F}(x, y) + \int_0^y \mathcal{F}(x, t)\mathcal{K}(y, t)^T\,dt\\ =&\sum_{n=0}^{\infty} c_n\varphi_n^o(x)\varphi_n^o(y)^T + \int_0^y \sum_{n=0}^{\infty} c_n\varphi_n^o(x)\varphi_n^o(t)^T\mathcal{K}(y, t)^T\,dt\\ =&\sum_{n=0}^{\infty} c_n\varphi_n^o(x)\left(\varphi_n^o(y)^T + \int_0^y \varphi_n^o(t)^T\mathcal{K}(y, t)^T\,dt\right)\\ =&\sum_{n=0}^{\infty} c_n\varphi_n^o(x)\left(\varphi_n^o(y) + \int_0^y \mathcal{K}(y, t)\varphi_n^o(t)\,dt\right)^T\\ =&\sum_{n=0}^{\infty} c_n\varphi_n^o(x)\varphi_n(y)^T.\end{aligned} \tag{11.3}$$

We consider the integral in the proposed transformation equation first:

$$\begin{aligned}\int_0^x \mathcal{L}(x, t)\mathcal{G}(t, y)\,dt =& -\int_0^x \mathcal{L}(x, t)\sum_{n=0}^{\infty} c_n\varphi_n(t)\varphi_n(y)^T\,dt\\ =& -\sum_{n=0}^{\infty} c_n\int_0^x \mathcal{L}(x, t)\varphi_n(t)\,dt\,\varphi_n(y)^T\\ =& -\sum_{n=0}^{\infty} c_n\mathcal{L}\varphi_n(x)\varphi_n(y)^T.\end{aligned}$$

Now, because

$$(I + \mathcal{L})\varphi_n = (I + \mathcal{L})(I + \mathcal{K})\varphi_n^o = \varphi_n^o,$$

we have, on adding $\mathcal{L}(x, y) + \mathcal{G}(x, y)$, that, for $0 \le y \le x \le \pi$,

$$\begin{aligned}&\mathcal{L}(x, y) + \mathcal{G}(x, y) + \int_0^x \mathcal{L}(x, t)\mathcal{G}(t, y)\,dt\\ &\quad=\mathcal{L}(x, y) - \sum_{n=0}^{\infty} c_n\varphi_n(x)\varphi_n(y)^T - \sum_{n=0}^{\infty} c_n\mathcal{L}\varphi_n(x)\varphi_n(y)^T\\ &\quad=\mathcal{L}(x, y) - \sum_{n=0}^{\infty} c_n\big(\varphi_n(x) + \mathcal{L}\varphi_n(x)\big)\varphi_n(y)^T\\ &\quad=\mathcal{L}(x, y) - \sum_{n=0}^{\infty} c_n\varphi_n^o(x)\varphi_n(y)^T = 0,\end{aligned}$$

which is what we set out to prove.