Today we finish the Baby Riesz note and go on to Chapter 3. We will start with an arbitrary continuous linear functional L, construct a map $\Lambda^+ : C[0,\,1] \to \mathbb{R}$ that is what we might call "positive linear" and satisfies $|Lf| \le \Lambda^+(|f|)$. We then extend this map to a continuous linear functional $\Lambda : C[0,\,1] \to \mathbb{R}$ and use it to show that the method of proof we used in the Special Case can be adapted to get a function of bounded variation that gives us a representation of L as a Riemann-Stieltjes integral. But we will need to use Further Problem 3 along the way!

The bottom 5/8 or so of the third page constructs Λ^+. Any questions? Maybe there's an easier way to split the function ℓ that appears in the proof of additivity when the functions are non-negative!

The last 3 lines on that page start the extension to a continuous linear functional. The bottom line takes care of the continuity estimate, tho linearity has not yet been shown. The observation that $\Lambda(g) = \Lambda^+(g) \ge 0$ is important! I skipped some steps in the note! Any questions?

You should check on your own that

$$(-f)^+ = f^- \text{ and } (-f)^- = f^+!$$

the key to showing linearity is to rearrange the identity

$$(f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$$

as an equality between sums of non-negative terms.

Then we can use the additivity of Λ^+ on both sides...

We are still (actually) a long way from getting what we want. Nevertheless, trying to follow the proof in the Special Case is the way to start.

So we define (same way) the $a_n(x) := L(\varphi_n(x))$. Immediately, we run into problems. They are not increasing functions, nor are they (necessarily) non-negative. This is where our linear functional Λ enters as a useful tool. We define the functions $A_n(x)$ to be the analogs of the $a_n(x)$, but for Λ instead of L.

Then, we can use what we learned in the Special Case and show (bottom part of top half, 4th page) that the a_n have uniformly bounded variation, and are uniformly bounded. We are almost ready to apply Further Problem 3, to the positive and negative variations $P_n(x)$ and $N_n(x)$ of the $a_n(x)$, we just have to know that the P_n and the N_n are uniformly bounded. This follows from $V = P + N$.

In order for you to understrand this stuff, you now have to carefully go thru the *steps* in the Special Case argument and check that they work. Here is where I expect questions!!

Once you do that you'll have a function $\alpha(x)$ that does work to give a representation, but we always want to know whether there are other ones, and if so, how they differ from the one we found. The rest of the note is devoted to that question.

The Lemma is proved in such gory detail that I just ask that you read it and ask questions. Please ask questions!

The proof includes a useful trick for avoiding repetition of a proof to deal with reflection or reversal of inequalities.

The Exercise will soon become a Further Problem, and Exercises will be made up from the "last" paragraph.

It has occured to me that maybe we *can't* find a unique representing α, when I consider the example $L_0 + L_1$. So I need to think of a substitute. Or, maybe you can!

I am going to see whether I can pin down the possible cases that depend on the behavior of α at the endpoints.

For now, disregard the "last" paragraph.

On to Chap. 3

This is where we begin our study of the Lebesgue integral. We begin with *outer measure,* defined in terms of the natural idea of the "volume" of an n-dimensional interval

$$I := \{x \in \mathbb{R}^n : a_i \le x_i \le b_i, \quad i = 1, \ldots, n\},$$

where the numbers a_i and b_i each satisfy

$$-\infty < a_i \le b_i < +\infty.$$

We can also think of I as the Cartesian product of closed bounded intervals $[a_i, b_i]$. We define the *volume,* $v(I)$, of I to be the product of the lengths of the intervals $[a_i, b_i]$. In symbols,

$$v(I) := \prod_{j=1}^{n} (b_i - a_i)$$

Thus we *can* have intervals of zero volume. But we often exclude them!

It's important to remember that these intervals have edges parallel to the coordinate axes. An interval is, by definition a Cartesian product, viewed as a subset of $\mathbb{R}^n$. We cannot yet rotate them! That will come later, however.

The notes break here, because I'll be in class Sept 24 after all.