The proof of (3.16) contains another "Clearly," but this one is not so bad. First, tho, we set aside any I'_k that does not meet E_1, and set aside any I''_k that does not meet E_2. We know that no I'_k that we kept meets E_2, and no I''_k that we kept meets E_1. Thus no such I'_k or I''_k has to be "used twice!" We can now "put back" the I'_k and I''_k that we set aside. The equality in the displayed line uses (3.15).

The proof of (3.14) is deep. It features a reduction to the case of compact sets F, using the fact that $\mathbb{R}^n$ is *σ-compact,* meaning that $\mathbb{R}^n$ is the union of countably many compact sets. This works because the class of measurable sets is closed under countable unions. Thus we may concentrate on showing that a compact set F is measurable. The outer measure of F is finite (why?) so there exists an open set $G \supseteq F$ such that $|G|_e < |F|_e + \epsilon$. We know that $G \setminus F$ is open so there exist non-overlapping cubes Q_k such that $G \setminus F = \bigcup_k Q_k$. We have $|G \setminus F|_e \leq \sum_k |Q_k|_e$; we need a good estimate for $\displaystyle\sum_k |Q_k|_e = \sum_k |Q_k|$ (by (3.13)) $\displaystyle = \lim_{N\to\infty} \sum_{k=1}^{N} |Q_k|$. We return to $|G|_e < |F|_e + \epsilon$. This suggests looking at

$$G = F \cup G \setminus F = F \cup \bigcup_k Q_k \text{ and then at}$$

$$|G| = |G|_e \geq \left| F \cup \bigcup_{k=1}^{N} Q_k \right|_e .$$

Since F and $\bigcup_{k=1}^{N} Q_k$ are disjoint and compact we can apply (3.16), and this gives

$$|G|_e \geq |F|_e + \left| \bigcup_{k=1}^{N} Q_k \right|_e = |F|_e + \sum_{k=1}^{N} |Q_k| \text{ by (3.15).}$$

By our choice of G, $\sum_{k=1}^{N} |Q_k| < \epsilon$, $\sum_{k=1}^{\infty} |Q_k| \leq \epsilon$, so $|G \setminus F|_e \leq \epsilon$, and F is measurable.

I recommend that you "translate" the proof of (3.17) from "CE" form to "E^c" form, and remember De Morgan's laws.

(3.18) and (3.19) are now easy to prove. We now know that the class $\mathcal{L}$ of Lebesgue measurable sets is closed under complementation, countable unions, countable intersections and set differences.

Def: A class Σ of subsets of a set X is a σ-*algebra* if
(1) Σ is non-empty,
(2) $E \in \Sigma \Rightarrow E^c \in \Sigma$,
(3) $E_k \in \Sigma, \quad k = 1, 2, \ldots, n, \ldots \Rightarrow \bigcup_k E_k \in \Sigma$.

(easy) **Exercises:** Show that $\emptyset \in \Sigma, X \in \Sigma$, and that Σ is closed under countable unions and set differences.
Show that 2^X and $\{\emptyset, X\}$ are σ-algebras.
Verify (less easy) the statements made in the discussion after (3.20), in the paragraph beginning "Given…"

(3.20) holds, due to the Theorems proved before the definition of σ-algebra.

The *Borel σ-algebra* is introduced in the discuassion following (3.20) and leading up to Section 3. We won't have much occasion to use Borel sets, but they and the discussion may be needed by some of you in other contexts. *Probability and Measure,* 2nd ed., by Patrick Billingsley, ISBN 0-471-80478-9, sections 2 and 3 of Chapter 1 (and elsewhere) has a lot of material on the properties and methods of construction of σ-algebras, called σ-fields there.

Here are some definitions taken from pp 162 and 178:

A *measure* is an *extended* real-valued function $\mu : \Sigma \to \overline{\mathbb{R}}$ such that
(i) $0 \le \mu(E) \le +\infty$ for all $E \in \Sigma$ and
(ii) $\mu(\bigcup_k E_k) = \sum_k \mu(E_k)$ if the E_k are countable and pairwise disjoint.

A *measure space* is a triple (X, Σ, μ) consisting of a set, a σ-algebra Σ and a measure $\mu : \Sigma \to \overline{\mathbb{R}}$.

A measure space is *σ-finite* if there is a sequence of sets $E_k \in \Sigma$ such that (i) $\mu(E_k) < \infty$, (ii) $\bigcup_k E_k = X$.

We have shown that $(\mathbb{R}^n, \mathcal{L}, |\cdot|)$ is a σ-finite measure space. For us, $\mu(E) := |E|$.

Section 3 has six important results one one that is nearly so, despite the section title.

The first is a "dual" version of the definition of measurable set – still in topological terms: (3.22). Proof is easy and a good review.

(3.23) is one of the section-title results, very important! The proof relies more on σ-compactness than on σ-finiteness, and makes heavy use of (3.22) and (3.16).

The initial assumption that each E_k is bounded allows the entry of compactness. But no boundedness assumption is made on the union of the E_k! The approach here is again a reduction to a special case!

(3.24) is the "nearly important" result. The proof has a "clear" that is indeed clear (do you agree?).

(3.25) brings in a recurrent thing – we have to carefully guard against taking differences or ratios of infinities! Of course there are exceptions such as $(+\infty) - (-\infty) = +\infty \ldots$

(3.26) Here the avoidance of subtracting two positive infinities arises in a subtle way! Be sure you note well the example following the proof!

(3.27) is the "not necessarily measurable" version of (3.26)i; its proof relies heavily on the existence of measurable sets that contain the E_k and have the same outer measure, so that (3.26), which relies heavily on (3.23), can be used.

Section 4 has three important characterizations of measurability. Two rely on our topological definition, while (3.30), due to Carathéodory, does not. Thus (3.30) will be the guide to definitions of measurability in the *abstract* setting.