Section 3 has six important results one one that is nearly so, despite the section title.

The first is a "dual" version of the definition of measurable set – still in topological terms: (3.22). Proof is easy and a good review.

(3.23) is one of the section-title results, very important! The proof relies more on σ-compactness than on σ-finiteness, and makes heavy use of (3.22) and (3.16).

A special case: each E_k is bounded. Then given $\epsilon > 0$ we can find closed sets $F_k \subseteq E_k$ such that $|E_k \setminus F_k| < \epsilon/2^k$. The F_k are compact and disjoint. We note that therefore $|E_k| \leq |F_k| + |E_k \setminus F_k| < |F_k| + \epsilon/2^k$. Thus by (3.16)

$$\left|\bigcup_{k=1}^{K} F_k\right| = \sum_{k=1}^{K} |F_k| > \sum_{k=1}^{K} |E_k| - \sum_{k=1}^{K} \epsilon/2^k > \sum_{k=1}^{K} |E_k| - \epsilon.$$

Thus for every K,

$$\left|\bigcup_{k=1}^{\infty} E_k\right| \geq \left|\bigcup_{k=1}^{K} E_k\right| \geq \left|\bigcup_{k=1}^{K} F_k\right| > \sum_{k=1}^{K} |E_k| - \epsilon.$$

Since $\epsilon > 0$ is arbitrary we conclude that $|\bigcup_{k=1}^{\infty} E_k| \geq \sum_{k=1}^{K} |E_k|$, so that $|\bigcup_{k=1}^{\infty} E_k| \geq \sum_{k=1}^{\infty} |E_k|$. We already know the reverse inequality holds [(3.12)], so this case is done.

The remaining case is reduced to this one by using the idea that $\mathbb{R}^n$ is "sigma-bounded," or that $\mathbb{R}^n$ is the union of countably many bounded sets.

We let Q_j denote the cube with center 0 and edge 2^j. Then the sets Δ_j given by $\Delta_1 := Q_1$, $\Delta_j := Q_j \setminus Q_{j-1}$ if $j > 1$, are bounded, disjoint and measurable, and their union is all of $\mathbb{R}^n$.

Now the special case applies to the sets $E_{kj} := E_k \cap \Delta_j$, arranged in a sequence, so we have

$$\left| \bigcup_{k,j} E_{kj} \right| = \sum_{k,j} |E_k \cap \Delta_j| = \sum_k \sum_j |E_k \cap \Delta_j| = \sum_k |E_k|.$$

A great deal is hidden in the last line! The first equality is not surprising – it follows from the special case. The second follows from the fact that series of non-negative numbers, taken over *any* index set, can be summed in *any* order, with the same result. This holds even when the sum is $+\infty$! The last equality again uses the special case. The proof is done.

(3.24) is the "nearly important" result. The proof has a "clear" that is indeed clear, by (3.12). Then we can get the other inequality using the interiors of the intervals, which are now disjoint.

(3.25) brings in a recurrent theme – we have to carefully guard against taking differences or ratios of infinities! Of course there are exceptions such as $(+\infty) - (-\infty) = +\infty \ldots$

(3.26) Here the avoidance of subtracting two positive infinities arises in a subtle way! Be sure you note well the example following the proof!

(3.27) is the "not necessarily measurable" version of (3.26)i; its proof relies heavily on the existence of measurable sets that contain the E_k and have the same outer measure, so that (3.26), which relies heavily on (3.23), can be used.

Section 4 has three important characterizations of measurability. Two rely on our topological definition, while (3.30), due to Carathéodory, does not. Thus (3.30) will be the guide to definitions of measurability in the *abstract* setting.

(3.28)i says that every measurable set can be obtained from a G_δ set by removing a set of measure zero. If E is measurable, we choose $G_k \supseteq E$ such that $|G_k \setminus E| < 1/k$. Then

$$H := \bigcap_k G_k \supseteq E, \quad |H \setminus E| \le \inf_k 1/k = 0,$$

so $E = H \setminus (H \setminus E)$ has the representation we want. And, if $E = H \setminus Z$, then E is measurable.

(3.28)ii says what happens if we use complements: if E is measurable, then $E^c = H \setminus Z = H \cap Z^c$, so $E = H^c \cup Z$. By De Morgan, H^c is an F_σ. Converse: immediate.

(3.29) will be Further Problem 4.

(3.30) characterizes measurability in terms of outer measure. Thus if we have obtained an "outer measure" somehow, we can define "measurable set" using (3.30).

The "easy" half of the proof: assume E is measurable. We look at an arbitrary set A and find a G_δ $H \supseteq A$ such that $|H| = |A|_e$. Since H and E are measurable and $|H| = |A|_e$, $|A|_e = |H| = |H \cap E| + |H \cap E^c| \geq |A \cap E|_e + |A \cap E^c|_e$. We already know the opposite inequality.

Next we assume that the equation

$$|A|_e = |A \cap E|_e + |A \cap E^c|_e = |A \cap E|_e + |A \setminus E|_e$$

is true for all sets $A \subseteq \mathbb{R}^n$. There *is* an easy case; it's the reduction to that case that's delicate.

The case: $|E|_e < +\infty$. Choose a G_δ $H \supseteq E$ such that $|H| = |E|_e$. Then, treating H as A,

$$(|E|_e =)|H| = |H \cap E|_e + |H \cap E^c|_e = |E|_e + |H \cap E^c|_e.$$

Hence (since $|E|_e < +\infty$) $|H \cap E^c|_e = 0$, so $Z := H \setminus E$ is measurable, since it has outer measure zero, and $E = H \setminus Z$. Thus E is measurable in this case.

In case $|E|_e < +\infty$ we want to find a suitable set H, *not necessarily a* G_δ, but measurable, and a set Z of measure zero, such that $E = H \setminus Z$. Thus will show that E is measurable. In fact, we'll find a set H that is a $G_{\delta\sigma}$.

We let $E_k := E \cap B_k(0)$, so that $|E_k|_e < +\infty$. We then find a G_δ $H_k \supseteq E_k$ with $|H_k| = |E_k|_e$. Then

$$|E_k|_e = |H_k| = |H_k \cap E|_e + |H_k \cap E^c|_e \geq |E_k|_e + |H_k \setminus E|_e.$$

The last step was the tricky one! We stick to what we were given, instead of trying to show that the equation we started with holds for each E_k!

We have shown that each $|H_k \cap E^c|_e = 0$. Now we put $H := \bigcup_k H_k$, which is a $G_{\delta\sigma}$, hence measurable, $E \subseteq H$, and $Z := H \setminus E = \bigcup_k (H_k \setminus E)$ has measure zero, being a countable union of set of measure zero. The since $E = H \setminus Z$, E is measurable.

We will skip (3.31) and (3.32).

Please ignore the first two sentences in Section 5. We will prove (3.33), *then* we will prove (3.10), (3.34) and (3.35) without using the material on page 7.

The approach to proving (3.33) is to look at *two* special cases, and combine them using unions. The cases are:
(1) Show continuous maps send closed sets to F_σ's;
(2) Show Lipschitz maps preserve sets of measure zero.

To do (1) we use the fact that a continuous map sends compact sets to compact sets, and that closed sets are countable unions of compact sets. Lipschitz maps are continuous.

For (2), we note that $\mathrm{diam}(E) = d \Rightarrow \mathrm{diam}(T(E)) \leq Cd$, where C is the Lipschitz constant of T. Thus a cube with edge e is mapped into a ball of radius at most $Ce\sqrt{n}/2$, hence a cube of edge at most $(C\sqrt{n})e$. We can replace a covering by intervals by one consisting of cubes having total volumeat most $(1+\epsilon)$ times the original total volume…