Conclusion of the Laurent Series Theorem

So far we have shown that if f is analytic on the closed annular region $0 < R_1 \leq |z - z_o| < R_2$, C_1 is the circle $|z - z_o| = R_1$ and C_2 is the circle $|z - z_o| = R_2$ then for every z with $R_1 < |z - z_o| < R_2$,

$$(1) \qquad f(z) = \sum_{n=-\infty}^{-1} (z - z_o)^n \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z_o)^{n+1}}\, d\zeta + \sum_{n=0}^{\infty} (z - z_o)^n \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z_o)^{n+1}}\, d\zeta.$$

There is a more pleasant formula – we get it by using the Deformation of Contours Principle (Corollary 2, Section 46, p 151) that says: given a function $F(z)$ analytic in a domain D, a simple closed curve $C_2 \subseteq D$ and a simple closed curve C_1 enclosed by C_2,

$$(2) \qquad \int_{C_1} F(z)\, dz = \int_{C_2} F(z)\, dz.$$

The argument we used in class to get the Laurent Series calculation started (it involved three contours) was an application of the Cauchy-Goursat Theorem. The same argument is used to show that (2) is true.

Now we can return to (1). We know (by compactness) that there is a slightly smaller disc, C_1', enclosed by C_1 and a slightly larger disc, C_2', that encloses C_2 so that f is analytic in the annular domain between C_1' and C_2'. Then C_2 and C_1 satisfy the conditions we need to apply the Deformation of Contours Principle. Thus we can use either contour in (1) and just call it C, to get:

$$(3) \qquad f(z) = \sum_{n=-\infty}^{\infty} (z - z_o)^n \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta - z_o)^{n+1}}\, d\zeta.$$

Even more is true! We can use the Deformation of Contours Principle again, using any other simple closed curve in our original closed annular region, to see that (3) holds for an arbitrary simple closed curve C in our original closed annular region. This is *Laurent's Theorem,* in Section 55, pp 190-191.

A uniqueness theorem for analytic functions

Theorem: *If f and g are analytic in a domain D and $f(z_n) = g(z_n)$ on a sequence of points $z_n \neq z_o$ lying in D that has limit $z_o \in D$, then $f(z) = g(z)$ for all $z \in D$.*

A less general theorem appears in two places in the text: the lemma in Section 26 (p 80) and Theorem 3 in Section 68 (p 241). We will prove this in two stages. First, we know that D contains a disc with center z_o and largest radius $R > 0$. If $R = +\infty$ then D is all of $\mathbb{C}$ and there will only be one stage. That said, we'll behave as though $0 < R < \infty$. We'll just consider the z_n with $|z_n - z_o| < R$ and prove that $h(z) := f(z) - g(z)$ is identically zero in that disc. Then we'll show that if z_1 is an arbitrary point in D then $h(z) = 0$ everywhere in a finite chain of overlapping discs that connects z_1 to z_o. Since this can be done for an *arbitrary* $z_1 \in D$ it can be done for all $z \in D$. This will show that $f(z) = g(z)$ for all $z \in D$.

First Stage

We know there exist coefficients c_n such that $h(z) = f(z) - g(z) = \sum_{n=0}^{\infty} c_n (z - z_o)^n$. Since h is continuous at z_o, $c_0 = h(z_o) = 0$. It is then also true that $h'(z_o) = \lim_{n\to\infty} \frac{h(z_n) - h(z_o)}{z_n - z_o} = 0$ since all the numerators are zero. Thus $c_1 = h'(z_o) = 0$. Next we will show that $h_2(z) := \sum_{n=0}^{\infty} c_{n+2}(z - z_o)^n$ has radius of convergence R' at least R. Before we do that, let's notice that $h_2(z_o) = c_2$ and that $h_2(z) = \frac{h(z)}{(z - z_o)^2}$ when $z \in D$ and $z \neq z_o$. The radius of convergence R' of the series for h_2 is the reciprocal of $\limsup_{n\to\infty} |c_{n+2}|^{1/n}$ and this is equal to the supremum of the set of all the numbers L that are limits of subsequences of $\{|c_{n+2}|^{1/n}\}$. To show that $R' \geq R$ we want to show that $\limsup_{n\to\infty}|c_{n+2}|^{1/n} \le \limsup_{n\to\infty}|c_n|^{1/n}$. We can write $|c_{n+2}|^{1/n} = (|c_{n+2}|^{1/(n+2)})^{(n+2)/n}$. So we want to know that $a_n^{p_n} \to a^p$, if $a_n \to a$ and $p_n \to p$. Because we will need it later, let's replace $n+2$ by $n+m$, where m is a positive integer. We next apply the following Lemma, to be proved later.

Lemma 1: *Let $\{a_k\}$ be a sequence of nonnegative numbers that has limit $a \in [0, \infty]$ and let $\{p_k\}$ be a sequence of real numbers that converges to the real number p. If $0 < a < \infty$, then $a_k^{p_k} \to a^p$. If $a = 0$ and $p > 0$ then $a_k^{p_k} \to a^p = 0$. If $a = \infty$ and $p > 0$ then $a_k^{p_k} \to \infty$.*

To apply this, we suppose that $|c_{n_k+m}|^{1/n_k} \to L'$ as $k \to \infty$. We put $a_k := |c_{n_k+m}|^{1/n_k}$ and $p_k = n_k/(n_k+m)$. Then $a_k \to L'$ and $a_k^{p_k} = |c_{n_k+m}|^{1/(n_k+m)}$. Since $p_k \to 1 > 0$, $|c_{n_k+m}|^{1/(n_k+m)} = a_k^{p_k} \to L'$ by the Lemma. Therefore every subsequential limit of the $|c_{n+m}|^{1/n}$ is also a subsequential limit of the $|c_n|^{1/n}$ sequence. In other "words,"

$$SL' := \{L' : L' \text{ is a subsequential limit of the } |c_{n+m}|^{1/n}\} \subseteq \{L : L \text{ is a subsequential limit of the } |c_n|^{1/n}\} =: SL.$$

Hence
$$\frac{1}{R'} = \limsup_{n\to\infty}|c_{n+m}|^{1/n} \le \limsup_{n\to\infty}|c_n|^{1/n} \le \frac{1}{R} \text{ and so } R' \ge R.$$

Therefore the series for $h_2(z)$ converges absolutely for $|z - z_o| < R$. But then h_2 is, in particular, continuous at z_o. Thus as we already noticed, $c_2 = \lim_{z\to z_o} h_2(z) = \lim_{z\to z_o} \frac{h(z)}{(z-z_o)^2} = \lim_{n\to\infty} \frac{h(z_n)}{(z_n - z_o)^2} = 0$ since each $h(z_n) = 0$. But then $c_2 = h''(z_o)/2 = 0$. For the same reasons as before, $h_2'(z_o) = c_3 = 0$. We can now make an induction proof that $c_n = 0$ for all n and thus $h(z) = 0$ if $|z - z_o| < R$. Moreover, for all $z \in D$ such that $|z - z_o| \le R$ we must have $h(z) = 0$, by continuity. We have completed the First Stage, except for the proof of the Lemma. That proof is in an Appendix to this Note.

Second Stage

Suppose that $z_1 \in D$. If $|z_1 - z_o| \le R$ we already know that $h(z_1) = 0$. Thus we suppose that $|z_1 - z_o| > R$. There is a polygonal path $P : [a, b] \to \mathbb{C}$, such that $P(a) = z_o$ and $P(b) = z_1$. We will move from z_o to z_1 in a finite number of steps along P. The first step will be on the boundary of the disc we used in Stage One, at a point $P(t_1)$. We need to find t_1 with $|P(t_1) - z_o| = R$. We use Completeness.

We define a set $\mathcal{O}_1 := \{t \in [a, b] : |P(t) - z_o| \ge R\}$. This is the set of all $t \in [a, b]$ such that $P(t)$ lies outside

the disc with center z_o and radius R that we used in Stage One. The set $\mathcal{O}_1$ is nonempty because b is in it. The set $\mathcal{O}_1$ is bounded below by a. Hence $t_1 := \inf \mathcal{O}_1$ exists. Let us note some properties of t_1: (1) if $t < t_1$ then $|P(t) - z_o| < R$ (Why?) Thus $|P(t_1) - z_o| \le R$ by continuity from below. (2) $|P(t_1) - z_o| = R$. To verify this, suppose $\epsilon > 0$ is given. If $s > t_1$ then there exists $t \in [a, b]$ such that $t < s$ and $|P(t) - z_o| \ge R$. So, we choose $s_o > t_1$ such that $|P(s) - P(t_1)| < \epsilon$ whenever $s_o > s > t_1$. We then let $s = (s_o + t_1)/2 > t_1$ and find $t \in [a, b]$ such that $t < s$ and $|P(t) - z_o| \ge R$. Thus $t \in \mathcal{O}_1$. Hence

$$R \le |P(t) - z_o| = |P(t) - P(t_1) + P(t_1) - z_o| \le |P(t_1) - z_o| + |P(t) - P(t_1)| < |P(t_1) - z_o| + \epsilon,$$

so $|P(t_1) - z_o| > R - \epsilon$. It follows (Why?) that $|P(t_1) - z_o| \ge R$, so $|P(t_1) - z_o| = R$, as claimed.

This gives us a point on the path P that is also on the boundary of our disc from Stage One. Let us call that disc B_0. We can now let R_1 be the radius of the largest (open!) disc with center $P(t_1)$ that is contained in D. We will call this disc B_1. We can now select a sequence $\{\tau_{1n}\}$ with $a \le \tau_{1n} < t_1$ such that $\tau_{1n} \to t_1$. Here is such a sequence: $\tau_{1n} = \frac{1}{n}a + \frac{n-1}{n}t_1$. We then put $z_{1n} = P(\tau_{1n})$ and we notice that $h(z_{1n}) = 0$. We can now re-use the argument in Stage One, with $p_1 := P(t_1)$ in place of z_o and with z_{1n} in place of z_n. That argument now shows that $h(z) = 0$ for all $z \in D$ such that $|z - p_1| = |z - P(t_1)| \le R_1$. We have shown now that $h(z) = 0$ in the union of the two overlappling discs B_0 and B_1. By continuity $h(z) = 0$ as well on the parts of their boundaries that lie in D.

Next we will find a new point on P, $P(t_2)$, with $t_2 > t_1$ such that $p_2 := P(t_2)$ lies on the boundary of B_1. To do this we go through the argument from the beginning of Stage Two through the last paragraph. But we use p_1

in place of z_o, R_1 in place of R, t_1 in place of a, seek a point t_2 instead of t_1 and define a set $\mathcal{O}_2$ in place of $\mathcal{O}_1$. We also use s_1 instead of s_o. I suggest you copy the first part of the Second Stage, making all the indicated replacements.

This gives us a process that starts with a point p on P (our p was $p_1 = P(t_1) \in D$) and on the boundary of a disc B contained in D and constructs a new (largest!) disc B' contained in D with center p, and another point p', *further along* P *than* p, and on the boundary of B'. We can start the process over again, starting with p' instead of p. This is illustrated in the text in Figure 69 in Section 50, p 170. But we only start the process again if $|z_1 - p'| > R'$, where R' is the radius of B'. This is so because if $|z_1 - p'| \le R'$, then z_1 is in the closure of B'. That means $h(z_1) = 0$, which we are seeking to prove.

We imagine constructing more and more discs: $B_2, B_3, \cdots, B_k, \cdots$, using our process, that "step" along P toward z_1. The problem is that perhaps the discs will "pile up" at some point along P and never actually reach z_1. Thus our next task is to show that this cannot happen, and that after only a finite number of steps z_1 will be in one of the constructed discs or on its boundary, so that $h(z_1) = 0$ will be shown to be true.

We will show (1) that there is a $\delta > 0$ such that for every point $P(s)$ on P, the disc with radius δ and center $P(s)$ is contained in D and (2) that the points $p_n := P(t_n)$ and $p_{n+1} := P(t_{n+1})$ satisfy $|p_n - p_{n+1}| \ge \delta$. This will allow us to move from z_o to z_1 in steps that are large enough to prevent us from converging to a limit before we reach z_1. Showing this last statement is true will essentially complete the proof.

We need another Lemma, whose proof will again be relegated to the Appendix. The Lemma asserts something about the distance from a point $P(t)$ to the complement of D. We defined this in class:

$$\text{dist}\,(P(t),\, D^c) := \inf\{|P(t) - w| : w \in D^c\}.$$

Lemma 2: *$d(t) := \text{dist}\,(P(t),\, D^c)$ takes on a positive minimum value δ on $[a, b]$.*

Our application of this is that if we pick a point $P(T)$ on P then the largest disc with center $P(T)$ that is contained in D has radius at least δ. The Lemma shows that (1) is true.

To show that (2) is true we need to know something about the lengths of polygonal paths. Our path P consists of *finitely many* line segments connected at their endpoints to give a continuous contour. The length of P is defined to be the sum of the segment lengths, and segment length is defined to be the distance between the endpoints of the segment. If w_1 and w_2 are the endpoints of the segment, the length of the segment is $|w_1 - w_2|$. The part of P that lies between two points $P(s_1)$ and $P(s_2)$ on P is a "partial path" $P_{s_1 s_2} : I \to \mathbb{C}$, where $I = [s_1, s_2]$ and $P_{s_1 s_2}(t) = P(t)$ if $s_1 < s_2$ and $I = [s_2, s_1]$ if $s_1 > s_2$ and then $P_{s_1 s_2}(t) = P(s_1 + s_2 - t)$. Thus $P_{s_1 s_2}$ is a polygonal path and its length is the sum of the lengths of some of the segments of P or parts of segments of P. By the triangle inequality (perhaps applied several times), the length of $P_{s_1 s_2}$ is greater than or equal to $|P(s_1) - P(s_2)|$.

For example, the length of P_{a,t_1} is at least $|P(a) - P(t_1)| = |z_o - P(t_1)| = |z_o - p_1| = R \ge \delta$ and the length of P_{t_1,t_2} is at least $|P(t_2) - P(t_1)| = |p_2 - p_1| = R_1 \ge \delta$. Since P_{a,t_1} lies in B_0 (except for p_1) and P_{t_1,t_2} is a "later" part of P than P_{a,t_1}, the length of the path P_{a,t_2} is the sum of the lengths of P_{a,t_1} and P_{t_1,t_2} and so the length of P_{a,t_2} is at least 2δ. We can add more and more partial paths $P_{t_k,t_{k+1}}$ and each of these has length at least $|P(t_{k+1}) - P(t_k)| = |p_{k+1} - p_k| = R_k \ge \delta$. Hence the length of P_{a,t_n} is at least $n\delta$. Since the length of P is finite there exists an n such that $n\delta > L$, where L is the length of P. Hence in a finite number of steps we will "use up" all of P and this means that $z_1 = P(b)$ lies in one of our B_n's (or in the closure of one of them), so $h(z_1) = 0$ and this (except for the proofs of the Lemmas) finishes the proof.

Appendix

Lemma 1: *Let $\{a_k\}$ be a sequence of nonnegative numbers that has limit $a \in [0, \infty]$ and let $\{p_k\}$ be a sequence of real numbers that converges to the real number p. If $0 < a < \infty$, then $a_k^{p_k} \to a^p$. If $a = 0$ and $p > 0$ then $a_k^{p_k} \to a^p = 0$. If $a = \infty$ and $p > 0$ then $a_k^{p_k} \to \infty$.*

Proof: Suppose $0 < a < \infty$. Then $b_k := \log a_k^{p_k} = p_k \log a_k \to p \log a$ so $a_k^{p_k} = e^{b_k} \to e^{p \log a} = a^p$.

In the other two parts of Lemma 1, $p > 0$. Thus there is K so large that for $k \ge K$, $p_k > p/2$.

Next suppose $a = 0$ and let $\epsilon > 0$ be given. Then there is K' so large that if $k \geq K'$, $a_k < \epsilon^{2/p}$. Thus if $k \geq \max\{K, K'\}$, $a_k^{p_k} \leq a_k^{p/2}$ because when $x \in [0, 1)$ and $0 < \alpha < \beta$, $x^\alpha \geq x^\beta$ (higher positive powers of numbers less than one are smaller). Hence for these k, $a_k^{p_k} \leq a_k^{p/2} < (\epsilon^{2/p})^{p/2} = \epsilon$ since x^α is an increasing function of x on $(0, \infty)$ when $\alpha > 0$. here we used $\alpha = p/2$.

Finally suppose that $a = +\infty$. Let $R > 0$ be given, no matter how large. Then there is K' so large that $k \geq K'$ implies $a_k > R^{2/p}$. Thus if $k \geq \max\{K, K'\}$, $a_k^{p_k} > a_k^{p/2}$ because when $x \in (1, \infty)$ and $0 < \alpha < \beta$, $x^\alpha < x^\beta$ (higher positive powers of numbers larger than one are larger). For these k, $a_k^{p_k} > a_k^{p/2} > (R^{2/p})^{p/2} = R$. This is what we mean by $a_k^{p_k} \to \infty$. This completes the proof of Lemma 1.

Lemma 2: *$d(t) := \mathit{dist}\,(P(t), D^c)$ takes on a positive minimum value δ on $[a, b]$.*

Proof: We first show that $d(t)$ is continuous, then invoke the Theorem that continuous real-valued functions on compact sets *take on* maximum and minimum values.

We have $d(t) = \text{dist}\,(P(t), D^c) = \inf\{|P(t) - w| : w \in D^c\}$. Let $\epsilon > 0$ be given. With both of s and t in $[a, b]$ and w arbitrary but in D^c we have

$$d(t) \leq |P(t) - w| \leq |P(t) - P(s) + P(s) - w| \leq |P(t) - P(s)| + |P(s) - w|.$$

Since $d(s) + \epsilon > d(s)$ we know there exists $w \in D^c$ such that $|P(s) - w| < d(s) + \epsilon$. From the last line we have $d(t) \leq |P(t)-P(s)|+|P(s)-w| < |P(t)-P(s)|+d(s)+\epsilon$. Hence $d(t)-d(s) < |P(t)-P(s)|+\epsilon$. We now use the same argument (we will get a different w in it) with the rôles of s and t reversed to get that $d(s)-d(t) < |P(s)-P(t)|+\epsilon$. That is, $d(t) - d(s) > -|P(s) - P(t)| - \epsilon/2$. This shows that $|d(t) - d(s)| < |P(t) - P(s)| + \epsilon$. This is true for every $\epsilon > 0$, so we have shown that $|d(t) - d(s)| \leq |P(t) - P(s)|$. Since $|P(t)|$ is continuous because $P(t)$ is, so is $d(t)$. There is some $T \in [a, b]$ such that $d(T) \leq d(t)$ for every $t \in [a, b]$, and we set $\delta := d(T) > 0$. This completes the proof of Lemma 2.