Ask! Indicate your approach! Show your work! Good Luck! There are 8 problems, 5 pages, and 100 points.

(1) [15] Find the general solution of $2y' = -y^3$.

First assume that $y \neq 0$. Then

$$2\frac{dy}{dx} = -y^3 \Rightarrow \frac{-2dy}{y^3} = dx \Rightarrow \int \frac{-2dy}{y^3} = \int dx \quad [4]$$

$$\Rightarrow \frac{1}{y^2} = x + c \quad [3] \Rightarrow y^2 = \frac{1}{x+c} \Rightarrow y = \pm\frac{1}{\sqrt{x+c}}, \quad \text{where } x + c > 0; \quad [2].$$

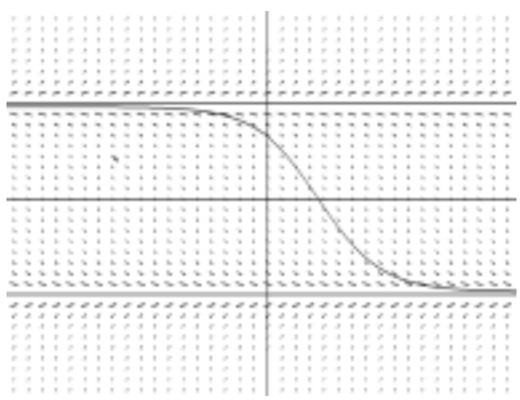
Also, $y(x) = 0$ is a solution [3]. Hence the general solution is:

$$y(x) = \frac{1}{\sqrt{x+c}}, \quad \text{where } x + c > 0, \quad \text{or } y(x) = -\frac{1}{\sqrt{x+c}}, \quad \text{where } x + c > 0, \quad \text{or } y(x) = 0 \quad [3].$$

(2) [10] Find the equation for the orthogonal trajectories: $4x + y = c$. **Be sure you make your method clear!**

$y = c - 4x$, $y' = -4 \Rightarrow$ the slope for the orthogonal trajectioies is $-\frac{1}{-4} = 1/4$. Thus the DE satisfied by the OT is $\frac{dy}{dx} = \frac{1}{4}$ [5] $\Rightarrow \int dy = \int \frac{dx}{4} \Rightarrow \left[\ y = \frac{1}{4}x + d, \text{ for an arbitrary constant } d\right]$ [5].

(3) [15] Sketch the slope field for the DE $y' = y^2 - 9$ **and an approximate solution curve for the IVP** $y' = y^2 - 9$, $y(0) = 2$.



Equilibrium solutions [2] each; slopes above, below them, [2] each; between them, [3]; the curve, [4].

(4) [10] Find four solutions of the IVP $y' = 3xy^{1/3}$, $y(0) = 0$.

The hoped-for answer: $y = 0$, $y = x^3$, then: $y(x) = 0$ for $x < 0$, $y(x) = x^3$ for $x \geq 0$ and $y(x) = x^3$ for $x < 0$, $y(x) = 0$ for $x \geq 0$.

Getting only to the separable-equation set-up: [2]; getting only to $y^{2/3} = x^2 + c$ and finding that $c = 0$: [3]; getting only to x^3: [4]; getting only to $\pm x^3$: [6]; $y = 0$ added [2]; getting four solutions: [10].

(5) [10] Find the general solution of $y' + xy = x$ **and check it.**

The integrating factor is $I(x) = e^{\int x\,dx} = e^{x^2/2}$ [2]. Thus $\left(e^{x^2/2}y\right)' = xe^{x^2/2}$ [2].

Then

$$e^{x^2/2}y = \int xe^{x^2/2}\,dx \quad [1]$$

$$(u = x^2/2, \quad du = x\,dx)$$

$$= \int e^u\,du = e^u + c \quad [2].$$

Thus $y(x) = \frac{1}{e^{x^2/2}}\left(e^{x^2/2} + c\right)$ and $\left[\quad y(x) = 1 + ce^{-x^2/2} \quad\right]$ [1].

Check: $y'(x) + xy(x) = -cxe^{-x^2/2} + x + cxe^{-x^2/2} = x \quad \surd$ [2].

(6) [15] Given that $P > 0$, solve the IVP $y' = y(P-y)$, $y(0) = P/2$.

$\frac{dy}{dx} = y(P-y) \Rightarrow \frac{dy}{y(P-y)} = dx$. We can divide by $y(P-y)$ because $y(0) = P/2 \Rightarrow$ we stay away from the equilibrium solutions 0 and P.
Using partial fractions,

$$x + c = \int dx = \int \frac{dy}{y(P-y)} = \frac{1}{P}\left[\int \frac{dy}{y} + \int \frac{dy}{P-y}\right] \quad [3];$$

$$= \frac{1}{P}\left[\ln|y| - \ln|P-y|\right] = \frac{1}{P}\ln\frac{|y|}{|P-y|} \quad [4];$$

$$\Rightarrow e^{P(x+c)} = \frac{|y|}{|P-y|} \quad [3];$$

$$y(0) = P/2 \Rightarrow e^{Pc} = \frac{P/2}{P/2} = 1 \Rightarrow c = 0 \quad [3].$$

Thus $y = \frac{Pe^{Px}}{1+e^{Px}} = \frac{P}{1+e^{-Px}}$ is the solution to the IVP [2] (since by uniqueness $0 < y < P$).

(7) [15] Find the general solution of $y'' - y = 4xe^x$.

1) Auxilary equation: $r^2 - 1 = 0$; $r = \pm 1$; [3].

2) Solution to homogeneous equation is $y(x) = c_1e^x + c_2e^{-x}$; [4].

3) Particular solution has the form $y_p(x) = (A_0 + A_1x)xe^x$; [3].

4) $y_p'' - y_p = (2A_1 + 2A_0)e^x + 4A_1xe^x = 4xe^x \Rightarrow A_1 = 1,\ A_0 = -1$; [2].

5) $y = c_1e^x + c_2e^{-x} + (-1+x)xe^x$; [1].

(8) [10] Find a particular solution of $y'' - 2y' + y = \cos x$.

1) Auxilary equation: $r^2 - 2r + 1 = 0$; $r_1 = 1 = r_2$, double root; [2].

2) i is not a root of the auxilary polynomial, so the trial solution is $y_p = A_0\cos x + B_0\sin x$; [5].

3) $y_p' = -A_0\sin x + B_0\cos x$; $y_p = -A_0\cos x - B_0\sin x \Rightarrow A_0 = 0,\ B_0 = -1/2$; [3].