Ask! Indicate your approach! Show your work! Good Luck! There are 8 problems, 5 pages, and 100 points.

(1) [20] Find the general solution of $y'' + y' + 2y = -5\sin t$.

$A(r) = r^2 + 2r + 2 = 0 \Rightarrow r_{1,2} = \frac{-2\pm\sqrt{4-8}}{2} \Rightarrow r_{1,2} = -1 \pm i$, so $y_c(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$ [5].

$$\text{Trial solution} \quad \left.\begin{aligned} y_p(t) &= A\cos t + B\sin t \\ y_p'(t) &= -A\sin t + B\cos t \\ y_p''(t) &= -A\cos t - B\sin t \end{aligned}\right\} \ [5]$$

$$\begin{aligned} y_p''(t) + 2y_p'(t) + 2y_p(t) = -5\sin t &\Rightarrow \\ -A\cos t - B\sin t - 2A\sin t + 2B\cos t + 2A\cos t + 2B\sin t = -5\sin t &\Rightarrow \\ (-A + 2B + 2A)\cos t + (-B - 2A + 2B)\sin t = -5\sin t &\Rightarrow \\ (A + 2B)\cos t + (B - 2A)\sin t = -5\sin t &\Rightarrow \end{aligned}$$

$$\left.\begin{aligned} A + 2B &= 0 \\ -2A + B &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ B &= -1 \end{aligned} \quad [5].$$

Thus $y_p(t) = 2\cos t - \sin t$ [3], so $y_g(t) = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} + 2\cos t - \sin t$ [2].

(2) [10] Solve the system $x - y + z = 1,\ x + y - 5z = 1,\ -x + 3y - 2z = 3$.

The system is equivalent to $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -5 \\ -1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ [2]. The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -5 & 1 \\ -1 & 3 & -2 & 3 \end{pmatrix} \xrightarrow[A_{13}(1)]{A_{12}(-1)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 5 & 4 \end{pmatrix} \xrightarrow[A_{23}(-2)]{M_2(1/2)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 5 & 4 \end{pmatrix} \quad [5].$$

$$\text{Therefore} \quad \left.\begin{aligned} 5z &= 4 \\ y - 3z &= 0 \\ x - y + z &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} z &= 4/5 \\ y &= 12/5 \\ x &= 13/5 \end{aligned} \quad [3].$$

(3) [15] Reduce the matrix $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ **to RREF. What is the rank of this matrix?**

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & -1 & -1 \end{pmatrix} \overset{1}{\rightarrow} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -4 & -1 & -3 \\ 0 & -1 & -2 & -1 \end{pmatrix} \overset{2}{\rightarrow} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{3}{\rightarrow}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{4}{\rightarrow} \begin{pmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{5}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{where:}$$

1= $A_{12}(-1),\ A_{13}(-2),\ A_{14}(-1)$; **2=** $A_{23}(4),\ A_{24}(1)$; **3=** $M_3(1/7)$; **4=** $A_{21}(-2)$; **5=** $A_{31}(3),\ A_{32}(-2)$. [13]
RANK $= 3$ [2]; right answer [13]; the meaning of RREF alone, [4].

(4) [10] Find T^2, T^3 and T^4, given that $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. **Find** $(I+T)(I-T+T^2-T^3)$.

$$T^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

[2] each; $(I+T)(I-T^2+T^3-T^4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$ [4].

(5) [15] Find the determinant of $\begin{vmatrix} 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 3 & 4 & 2 \\ 2 & -1 & -1 & 2 \end{vmatrix}$.

There are many ways to approach this. The following one was not required. Row operation $A_{23}(-3)$ was done first, then $A_{32}(1)$, followed by expansion along row 2 (the sign pattern along row 2 is $-+-+$):

$$\begin{vmatrix} 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 3 & 4 & 2 \\ 2 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 2 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 2 & -1 & -1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & 3 & 1 \\ 0 & -2 & -1 \\ -1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 0 & -2 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 0 & 7 \\ 0 & -2 & -1 \\ -1 & -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 4 & -1 \\ 0 & -2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 7 \\ -2 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & -1 \\ -2 & -1 \end{vmatrix} = 14 + 2(-6) = 2.$$

In the expansion along row 2, $A_{31}(3)$ and $A_{31}(-1)$ were done, one on each determinant. Then each determinant was expanded along column 1, which has sign pattern $+-+$. Then the 2×2's were done by definition.

Method: [4]–[10], depending on methods shown and aptness. Up to [5] for organization, readability and answer.

(6) [15] Show that the functions $a+bx+cx^2+dx^3+ex^4$ (where a, b, c, d and e take on, as values, all possible quintuples of complex numbers) comprise a vector space of functions of x.

Two approaches:

(1): [4] Observe that "functions" comprise a vector space; [3] Show closed under addition; [3] Show closed under scalar multiplication; [5] **Cite** the subspace Theorem from §5.3.

(2): Verify the 10 properties of a vector space; citing by name OK; -[1.5] for each omission, rounded up (1 omission, -[2], 2 omissions -[3], etc.)

(7) [15] Find the inverse of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{A_{23}(-2)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{pmatrix} \xrightarrow{A_{21}(-2)} \begin{pmatrix} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow[A_{31}(3)]{A_{32}(-2)} \begin{pmatrix} 1 & 0 & 0 & 6 & -8 & 3 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}. \quad \text{Thus } A^{-1} = \begin{pmatrix} 6 & -8 & 3 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

For this question, if you convinced the grader that you understood the procedure and that you are able to carry it out you got [9]. If you made some arithmetic mistakes along the way you got [1] for each correct element in your answer (up to a maximum of [15], of course).

(8) [10] Find AB and BA, given that $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ **and** $B = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 2 & -3 \\ -1 & -1 & 2 \end{pmatrix}$. **Is it true that** $AB = BA$?

$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = BA$. [5] for each computation. So AB and BA are the same.

In fact, for two square matrices A and B, if $AB = I$ then BA is necessarily equal to I. In other words, if B is a right inverse for A then B is also a left inverse for A.