Math 2243, Fall '02, **Test 3:** Solutions Your Name: ____________________

Ask! Indicate your approach! Show your work! Good Luck! There are 8 problems, 5 pages, and 100 points.

(1) [10] Find a basis for the nullspace of $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 2 & 2 & 2 & 1 \end{pmatrix}$.

First, bring A to REF:

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & -1 \\ 2 & 2 & 2 & 1 \end{pmatrix} \overset{1}{\to} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \overset{2}{\to}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -3 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & -1 \end{pmatrix} \overset{3}{\to} \begin{pmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where:}$$

1=$A_{12}(-2)$, $A_{13}(-1)$, $A_{14}(-2)$; **2=**$M_1(-1)$, $A_{23}(-1)$; **3=**$M_3(-1/4), A_{34}(2), M_{-2}(-1/4)$ [5].

There are no zero rows in this REF of A, so the dimension of the nullspace of A is zero. Hence a basis for the nullspace of A is $\{\mathbf{0}\}$ (or, "empty basis," or "no basis" would be correct as well) [5].

Note: There are other ways to do this. For instance, observe that $\det A = 2 \neq 0$, hence the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, which means that the nullspace of A is $\{\mathbf{0}\}$.

(2) [10] Let V be the vector space of all differentiable real-valued functions defined on the interval $[0, 1]$ and let S be the set of all $f \in V$ such that $f(0) = f'(1)$. and $f'(0) = f(1)$. Express S in set notation and determine whether or not it is a subspace of V.

$$S = \{f \in C^1([0, 1]) : f(0) = f'(1),\ f'(0) = f(1)\} \quad [2].$$

(i) $\mathbf{0} \in S$: let f be the zero function on $[0, 1]$. Then $f(0) = f'(1)$ and $f'(0) = f(1)$
because all of these numbers are zero [2].

(ii) S is closed under scalar multiplication: let $f \in S$ and $\alpha \in \mathbb{R}$. Then
$(\alpha f)(0) = \alpha f(0)$ and $(\alpha f)'(1) = \alpha f'(1)$, so that $f(0) = f'(1) \Rightarrow \alpha f(0) = \alpha f'(1) \Rightarrow (\alpha f)(0) = (\alpha f)'(1)$;
$(\alpha f)'(0) = \alpha f'(0)$ and $(\alpha f)(1) = \alpha f(1)$, so that $f'(0) = f(1) \Rightarrow \alpha f'(0) = \alpha f(1) \Rightarrow (\alpha f)'(0) = (\alpha f)(1)$.
Hence $f \in S \Rightarrow \alpha f \in S$ for all $\alpha \in \mathbb{R}$ [3].

(iii) S is closed under vector addition: $f \in S$ and $g \in S$. Then
$(f+g)(0) = f(0) + g(0) = f'(1) + g'(1) = (f+g)'(1)$ and
$(f+g)'(0) = f'(0) + g'(0) = f(1) + g(1) = (f+g)(1)$ so $f + g \in S$.
Thus S is a subspace of V [3].

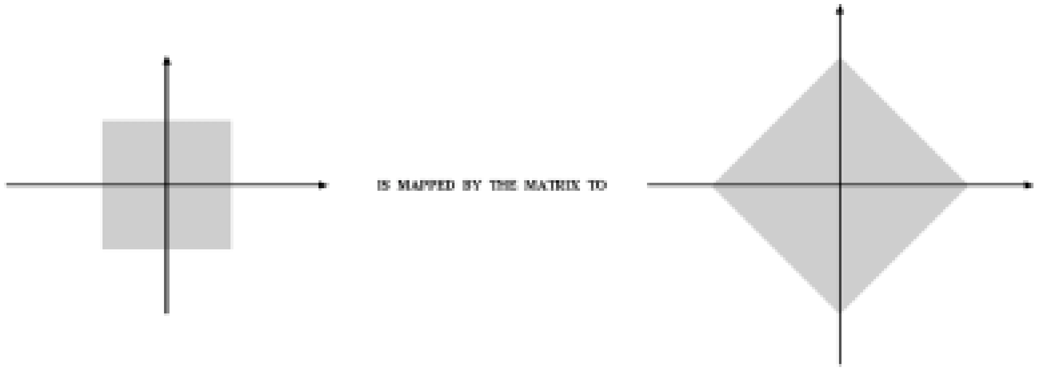
(3) [15] Sketch what the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
does to the rectangle determined by $(-1, -1), (1, -1), (1, 1)$.

Taking $A = (-1, -1)$, $B = (-1, -1)$, $C = (-1, -1)$, $D = (-1, -1)$, we get

$$A' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad [2], \quad B' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad [2];$$

$$C' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad [2], \quad D' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad [2],$$

and the sketch



[1] each.

(4) [10] Find the eigenvalues and the eigenvectors for the matrix $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

$|A - \lambda I| = 0 \Rightarrow (\lambda - 2)^2 - 9 = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 5, \quad \lambda_2 = -1$ [5].

For $\lambda_1 = 5$,

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [5].$$

For $\lambda_1 = -1$,

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{v} = s \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad [5].$$

(5) [15] Solve the VDE $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

The eigenvalues of A are $\lambda = 1$ and $\lambda = 5$. Corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is a matrix such that $S^{-1}AS = D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ [5].

Define $\mathbf{y} = S^{-1}\mathbf{x}$. Then $AS\mathbf{y} = A\mathbf{x}$ and if $\mathbf{y}' = S^{-1}\mathbf{x}'$ then $\mathbf{x}' = A\mathbf{x} \Rightarrow S^{-1}\mathbf{x}' = S^{-1}A\mathbf{x} \Rightarrow \mathbf{y}' = S^{-1}AS\mathbf{y} = D\mathbf{y}$. Thus we need to solve $\mathbf{y}' = D\mathbf{y}$, which is equivalent to (*explanation* not required)

$$\left.\begin{array}{r} y_1'(t) = y_1(t) \\ y_2'(t) = 5y_2(t) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y_1(t) = c_1 e^t \\ y_2(t) = c_2 e^{5t} \end{array}\right. \quad [5].$$

$\mathbf{y} = S^{-1}A\mathbf{x} \Rightarrow \mathbf{x} = S\mathbf{y}$. Hence

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^{5t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{5t} \\ -c_1 e^t + c_2 e^{5t} \end{pmatrix} \quad [5].$$

(6) [15] Let $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ **and** $\mathbf{q} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$. **Find all pairs of scalars c, d such that $\{\mathbf{p} + c\mathbf{q}, \mathbf{q} + d\mathbf{p}\}$ is LD.**

Two vectors are LD if and only if one of them is a constant multiple of the other. We have

$$\mathbf{p} + c\mathbf{q} = \begin{pmatrix} 1 + 2c \\ 2 \\ 1 + 2c \end{pmatrix} \quad \text{and} \quad \mathbf{q} + d\mathbf{p} = \begin{pmatrix} 2 + d \\ 2d \\ 2 + d \end{pmatrix} \quad [4].$$

Then

$$\mathbf{p} + c\mathbf{q} = \alpha(\mathbf{q} + d\mathbf{p}) \Rightarrow \alpha = \frac{2}{2d} = \frac{1 + 2c}{2 + d} \Rightarrow 4cd = 4 \Rightarrow cd = 1.$$

Thus the pairs $(c,\, d)$ with $cd = 1$ cause $\{\mathbf{p} + c\mathbf{q},\, \mathbf{q} + d\mathbf{p}\}$ to be LD [6].

(7) [15] Find all scalars C such that the matrix $\begin{pmatrix} 2 & C \\ 1 & 2 \end{pmatrix}$ is defective.

$\begin{vmatrix} 2-\lambda & C \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - C = \lambda^2 - 4\lambda + (4 - C) = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16 - 4(4-C)}}{2} \Rightarrow \lambda = 2 \pm \sqrt{C}$ [2].

$$\begin{pmatrix} 2 - (2 \pm \sqrt{C}\,) & C \\ 1 & 2 - (2 \pm \sqrt{C}\,) \end{pmatrix} = \begin{pmatrix} \mp\sqrt{C} & C \\ 1 & \mp\sqrt{C} \end{pmatrix},$$

$$\begin{pmatrix} \mp\sqrt{C} & C \\ 1 & \mp\sqrt{C} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \mp\sqrt{C}\,x + Cy & = 0 \\ x + \mp\sqrt{C}\,y & = 0 \end{matrix} \quad [2].$$

If $C = 0$, $\lambda = 2$ is an eigenvalue of multiplicity two, and then

$$\begin{pmatrix} \mp\sqrt{C} & C \\ 1 & \mp\sqrt{C} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 0 \text{ and } y \neq 0,$$

so there is essentially only one eigenvector, and so the matrix is defective if $C = 0$ [2].

If $C \neq 0$, there are two eigenvalues, so the matrix is not defective if $C \neq 0$ [4].

(8) [10] Find the eigenvalues of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, and find a real eigenvector of A. Is A defecteve? Why?

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)^2 + 1 = 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 1 = 2 - 3\lambda + 3\lambda^2 - \lambda^3.$$

Try roots of ± 2 and ± 1: 2 is a root [2]. By long division we find that $-(2 - 3\lambda + 3\lambda^2 - \lambda^3) = (\lambda - 2)(\lambda^2 - \lambda + 1)$ and then we find two more roots, $\dfrac{1 \pm i\sqrt{3}}{2}$ [4], so there are three different eigenvalues and so the matrix is not defective [4].

A real eigenvector (for $\lambda = 2$): we want a non-zero vector $(x,\, y,\, z)$ so that $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. We notice that each row adds to zero, so $(x,\, y,\, z) = (1,\, 1,\, 1)$ will work [5].