In **3.16,** we let $E = E^* \cup S$, where E^* is defined in **3.6** and

$$S := \begin{cases} \emptyset \text{ if no subsequence of } \{s_n\} \text{ has limit } +\infty \text{ and no subsequence has limit } -\infty; \\ \{+\infty\} \text{ if some subsequence of } \{s_n\} \text{ has limit } +\infty \text{ but no subsequence has limit } -\infty; \\ \{-\infty\} \text{ if some subsequence of } \{s_n\} \text{ has limit } -\infty \text{ but no subsequence has limit } +\infty; \\ \{-\infty, +\infty\} \text{ if some subsequence of } \{s_n\} \text{ has limit } +\infty \text{ and some subsequence has limit } -\infty. \end{cases}$$

Thus E is a set of *extended* real numbers, but E maight contain *only* real numbers. The point is that we are in the *context* of extended real numbers.

We set $s^* := \sup E$, denoted $\limsup_{n\to\infty} s_n$. We need to show:
(a) $s^* \in E$ *and*
(b) If $x > s^*$ *there exists* $N \in \mathbb{N}$ *such that* $n \geq N \Rightarrow s_n < x$.

Lemma: *If* $\{s_n\}$ *is not bounded above, then there exists a subsequence* $\{s_{n_i}\}$ *such that* $\lim_{i\to\infty} s_{n_i} = +\infty$.

Let us use the Lemma now, and have you give its proof later.

If $+\infty \in E$, some subsequence of $\{s_n\}$ has limit $+\infty$. If $+\infty \notin E$, then by the Lemma $\{s_n\}$ is bounded above.

There are now two possibilities: $s^* = -\infty$ and $s^* \in \mathbb{R}$.
If $s^* = -\infty$, then $E^* = \emptyset$, so that $\{s_n\}$ is bounded above but has no finite subsequential limits. Hence no subsequence converges, and we know that no subsequence has limit $+\infty$. Therefore every subsequence is unbounded below. To show that $s_n \to -\infty$ we let $R \in \mathbb{R}$ be given (presumably very negative). We have to show that there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow s_n < R$. Suppose not. Then for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $s_n \geq R$. For each $N \in \mathbb{N}$ we define $S_N := \{n \in \mathbb{Z}^+ : n \geq N \text{ and } s_n \geq R\}$. Our contradiction assumption amounts to the assumption that, for all $N \in \mathbb{N}$, $S_N \neq \emptyset$. Let us notice that $S_{N+1} \subseteq S_N$. We will use the Recursion Theorem to define a sequence that is bounded below. We let $Y := \mathbb{Z}^+$, and we define $H : Y \to Y$ as follows: $H(N) := \min S_N + N$. The minimum exists for each N because we assumed that each S_N is nonempty. We choose $n_1 := 1$. By the Recursion Theorem there exists a unique sequence $\{n_i\}$ such that $n_1 = 1$ and $n_{i+1} = H(n_i)$ for all $i \in \mathbb{Z}^+$. By the way we defined the sets S_N, $s_{n_i} \geq R$ for all $i \in \mathbb{Z}^+$. To show that $\{s_{n_i}\}$ is a subsequence we need to show that $n_{i+1} > n_i$ for all $i \in \mathbb{Z}^+$. Let $i \in \mathbb{Z}^+$ be arbitrary. Then

$$n_{i+1} = \min S_{n_i} + n_i \geq 1 + n_i > n_i.$$

We have found a subsequence $\{s_n\}$ of that is bounded below by R and this gives the contradiction we wanted. Thus $+\infty \in E$, so $s^* \in E$.

Next we suppose $s^* \in \mathbb{R}$. Then E^* must be nonempty and it must be true that $+\infty \notin E$. Moreover, E^* must be bounded above, for otherwise we would have $s^* = +\infty$ (why must E^* be nonempty?). Therefore $s^* = \sup E = \sup E^*$, so $s^* \in E^* \subseteq E$ by **3.7.**

Proof of the Lemma You can find a proof of the Lemma in the proof above, but for the "unbounded below" case.

We have finished the proof of (a).

To prove (b) we suppose that $x > s^*$. If we had $s^* = +\infty$, this would be false, so (b) would be true vacuously. We consider the case $s^* < +\infty$. There are two subcases: $s^* = -\infty$ and $s^* \in \mathbb{R}$.

In the first subcase we have shown that $s_n \to -\infty$, so (b) holds easily (are you sure?). In the second subcase, there exists $M \in \mathbb{R}$ such that $s_n \leq M$ for all $n \in \mathbb{N}$. This follows from the Lemma. This implies that every subsequential limit $t \leq M$. If $M < x$, we then have nothing to show. If $M \geq x$, we have to show that *eventually* $s_n < x$. To prove this we use the argument above that begins "We have to show that" (with R replaced by x) and ends just before the last sentence of its paragraph. This completes the proof of (b).

Next we need to show that s^* is the only extended real number that satisfies (a) and (b). To do so we assume that (also) $t \in \overline{\mathbb{R}}$ satisfies (a) and (b). We know that $t \leq s^*$ (why?). Suppose then that $t < s^*$. Then there exists $\tau \in \mathbb{R}$ such that $t < \tau < s^*$. Since $s^* \in E$ there exists a subsequence $s_{n_j} \to s^*$. Thus for some J, $j \geq J \Rightarrow s_{n_j} > \tau$ (why?). But $\tau > t$, so (b) is not satisfied for t. Contradiction! Hence $t = s^*$.