Introduction

We know that the Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$, diverges. In this note we examine the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ and two related series. We let $u_n := \frac{(-1)^{n+1}}{n}$, $v_n := \left(\frac{1}{2n-1}-\frac{1}{2n}\right)$ and we let w_n be the n-th number we get by taking the first two available terms u_n with *odd* subscripts, then the next available term u_n with an *even* subscript. This gives us

$$w_1 = 1,\ w_2 = \frac{1}{3},\ w_3 = -\frac{1}{2},\ w_4 = \frac{1}{5},\ w_5 = \frac{1}{7},\ w_6 = -\frac{1}{4},\ w_7 = \frac{1}{9},\ w_8 = \frac{1}{11},\ w_9 = -\frac{1}{6}\ \text{ and so on.}$$

Thus every term u_n appears as a w_n, exactly once, but in a different order. Every v_n is the sum of u_{2n-1} and u_{2n}, so the series $\sum_{n=1}^{\infty} v_n$ is constructed by combining adjacent terms of $\sum_{n=1}^{\infty} u_n$. Thus every u_n is accounted for among the v_m.

We will show that $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ both converge to the same limit, $L > 0$. It is natural to expect that $\sum_{n=1}^{\infty} w_n$ converges to L also, but this is not so! We will show that $\sum_{n=1}^{\infty} w_n$ does converge, to $\frac{3}{2}L$! Finally, we will prove in a different way that $\sum_{n=1}^{\infty} u_n$ converges, and find its sum, L.

Outline: proofs of convergence, using a Lemma

The convergence of a series is the same thing as convergence of its partial sums. We will use these notations for the partial sums of the three series:

$$U_n := \sum_{k=1}^{n} u_k, \qquad V_n := \sum_{k=1}^{n} v_k, \qquad W_n := \sum_{k=1}^{n} w_k.$$

First we show that $\{V_n\}$ converges to a positive limit L, using the Comparison Test and the p-test, with $p = 2$. Then we show that $V_n = U_{2n}$ so that $U_{2n} \to L$. We then state and use (proof later) a Lemma about how knowing that a subsequence (of a special kind!) converges *can* imply that the whole sequence converges. This will work to show that $U_n \to L$. Next we find a formula for what we get when we group the terms of $\sum_{n=1}^{\infty} w_n$ into *triples*, use this to get an identity involving certain v_m's that allows us to show that $W_{3n} \to 3L/2$. Once again we will use the Lemma to show that $W_n \to 3L/2$. Finally, we prove the Lemma. After all that is done, we'll prove in a different way that $\sum_{n=1}^{\infty} u_n$ converges, and find its sum.

Proofs of convergence, using a Lemma

We have

$$v_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{(2n-1)2n} = \frac{1}{4n^2-2n},$$

so each v_n is positive and thus $\{V_n\}$ is a strictly increasing sequence. Since $4n^2 - 2n \geq 4n^2 - 2n^2 = 2n^2$, we see that

$$v_n \leq \frac{1}{2n^2} < \frac{1}{n^2},$$

so by the Comparison Test and the p-test, with $p = 2$, $\sum_{n=1}^{\infty} v_n$ converges to some positive number L. That is, $V_n \to L$.

Each v_k uses two terms, u_{2k-1} and u_{2k}, from the series $\sum_{n=1}^{\infty} u_n$. Thus

$$V_n = \sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) = \sum_{k=1}^{2n} u_k = \sum_{k=1}^{2n}\frac{(-1)^{k+1}}{k} = U_{2n}, \quad \text{and therefore } U_{2n} \to L.$$

Here is the Lemma we need. The proof is on page 3. We'll go ahead and use it.

Lemma: *Suppose that* $\{x_n\}$ *is a sequence of real numbers and that for some integer* $J > 1$, *the subsequence* $\{x_{Jk}\}$ *converges to a limit* ℓ. *Suppose that, in addition,* $x_{Jk} - x_{Jk+i} \to 0$, *for each* i, $1 \leq i < J$. *Then* $\{x_n\}$ *converges to* ℓ.

In our first application of the Lemma, x_n will be U_n, J will be 2, and ℓ will be L. This makes one hypothesis translate to "$U_{2k} \to L$," which we have shown true. We have one more hypothesis to check: "$U_{2k} - U_{2k+1} \to 0$." Here, $U_{2k} - U_{2k+1} = -u_{2k+1} = -1/(2k+1) \to 0$. Thus the conclusion of the Lemma, that $\{U_n\}$ converges to L, is true. This means: $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = L$.

Next we have to connect the series $\sum_{n=1}^{\infty} w_n$ and the series $\sum_{n=1}^{\infty} v_n$, using the terms of the original series.

This takes some trial and error, and some guesswork. To begin let's write subscripts for w in one line, and below that put the subscripts of the terms from the u series that we use to follow the "next two odd, next even" rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	3	2	5	7	4	9	11	6	13	15	8	17	19	10	21	23	12	25	27	14	29	31	16

It's hard to see a single pattern here. That's because there are three patterns, mixed together. Let's write down the list in three pieces, counting by threes:

1	4	7	10	13	16	19	22
1	5	9	13	17	21	25	29

2	5	8	11	14	17	20	23
3	7	11	15	19	23	27	31

3	6	9	12	15	18	21	24
2	4	6	8	10	12	14	16.

Now we can see patterns. The first line in the first pair of lines consists of numbers that have the form $3k+1$, $k \geq 0$. The second line in the first pair of lines consists of numbers that have the form $4k+1$, $k \geq 0$.

The corresponding subscript formulas in the other two pairs of lines are $3k+2$ and $4k+3$, $k \geq 0$, for the second pair of lines and $3k+3$ and $2k+2$, $k \geq 0$, for the third pair of lines.

This gives us a tripartite formula for w_n:

$$w_n = \begin{cases} \frac{1}{4k+1} = \frac{3}{4n-1}, & \text{if } n = 3k+1, \quad k \geq 0; \\ \frac{1}{4k+3} = \frac{3}{4n+1}, & \text{if } n = 3k+2, \quad k \geq 0; \\ -\frac{1}{2k+2} = -\frac{3}{2n}, & \text{if } n = 3k+3, \quad k \geq 0. \end{cases}$$

This formula shows that $w_n \to 0$ as $n \to \infty$, information we will need later!

When we group the terms w_n into triples, we'll have $w_{3k+1} + w_{3k+2} + w_{3k+3}$, $\quad k \geq 0$. When we substitute in that $w_{3k+1} = u_{4k+1}$, $\;w_{3k+2} = u_{4k+3}$ and $w_{3k+3} = u_{2k+2}$ we get

$$w_{3k+1} + w_{2k+2} + w_{3k+3} = u_{4k+1} + u_{4k+3} + u_{2k+2} = \frac{1}{4k+1} + \frac{1}{4k+3} - \frac{1}{2k+2}.$$

Next we notice that $\frac{1}{2k+2} = \frac{2}{4k+4}$ so we can rewrite (subtracting and adding!):

$$\frac{1}{4k+1} - \frac{1}{4k+4} + \frac{1}{4k+3} - \frac{1}{4k+4} = \left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) + \left(\frac{1}{4k+2} - \frac{1}{4k+4}\right) + \left(\frac{1}{4k+3} - \frac{1}{4k+4}\right).$$

As $4k+1 = 2(2k+1) - 1$ and $4k+2 = 2(2k+1)$, $\left(\frac{1}{4k+1} - \frac{1}{4k+2}\right) = v_{2k+1}$. Similarly, $\left(\frac{1}{4k+3} - \frac{1}{4k+4}\right) = v_{2k+2}$. The other term, $\left(\frac{1}{4k+2} - \frac{1}{4k+4}\right)$ is different because its denominators are not adjacent integers. Moreover, both denominators are even. If we factor out 2 we then get $\left(\frac{1}{4k+2} - \frac{1}{4k+4}\right) = \frac{1}{2}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) = \frac{1}{2}v_{k+1}$. You should double check, *now,* to see whether this is right!! If all this is right, we have

$$w_{3k+1} + w_{3k+2} + w_{3k+3} = v_{2k+1} + v_{2k+2} + \frac{1}{2}v_{k+1}, \quad k \geq 0.$$

Now let's add from $k = 0$ to $k = n - 1$:

$$\sum_{k=0}^{n-1}(w_{3k+1} + w_{3k+2} + w_{3k+3}) = \sum_{k=0}^{n-1}(v_{2k+1} + v_{2k+2}) + \sum_{k=0}^{n-1} \frac{1}{2} v_{k+1}.$$

First, $\sum_{k=0}^{n-1}(w_{3k+1} + w_{3k+2} + w_{3k+3}) = W_{3n}$ (why?!).
Second, $\sum_{k=0}^{n-1}(v_{2k+1} + v_{2k+2}) = V_{2n}$ (why?!).
Third, $\sum_{k=0}^{n-1} \frac{1}{2} v_{k+1} = \frac{1}{2} V_n$ (why?!).

Therefore $W_{3n} = V_{2n} + \frac{1}{2} V_n \to 3L/2$ as $n \to \infty$.

To apply the Lemma, we need to check that $W_{3n} - W_{3n+1} \to 0$ and $W_{3n} - W_{3n+2} \to 0$. We noted earlier that $w_n \to 0$ as $n \to \infty$. Then $W_{3n} - W_{3n+1} = -w_{3n+1} \to 0$, and $W_{3n} - W_{3n+2} = -w_{3n+1} - w_{3n+2} \to 0$, so the Lemma's hypotheses are satisfied, hence $W_n \to 3L/2$.

Proof of the Lemma

Now we have to prove the Lemma. Here it is again:

Lemma: *Suppose that $\{x_n\}$ is a sequence of real numbers and that for some integer $J > 1$, the subsequence $\{x_{Jk}\}$ converges to a limit ℓ. Suppose that, in addition, $x_{Jk} - x_{Jk+i} \to 0$, for each i, $1 \le i < J$. Then $\{x_n\}$ converges to ℓ.*

Proof: Given $\epsilon > 0$ we use $\epsilon/2$ as the "epsilon" in the subsequence information we were given and so find integers K and K_i, $1 \le i \le J-1$, such that
(1) $k \ge K \Rightarrow |x_{Jk} - L| < \epsilon/2$
(2) $k \ge K_i \Rightarrow |x_{Jk} - x_{Jk+i}| < \epsilon/2$ (presumably a different K_i for each i, $1 \le i \le J-1$).

We now define $N := J \max\{K, K_1, \ldots, K_{J-1}\} + J$. Let's show that, if $n \ge N$, then $|x_n - \ell| < \epsilon$. We begin by writing $n = Jq + r$, where q is the quotient, and r the remainder we would get if we used long division to divide J into n. We know that $0 \le r < J$, so $Jq = n - r > n - J \ge N - J \ge JK + J - J = JK$. Thus $q \ge K$, so $|x_{Jq} - \ell| < \epsilon/2$. If, in $n = Jq + r$, we had $r = 0$ we would now know that for this particular n, $|x_n - \ell| < \epsilon/2 < \epsilon$. If $0 < r < J$, we have (arguing much as before) $Jq = n - r > n - J \ge N - J \ge JK_r + J - J = JK_r$. Thus $q \ge K_r$, so $|x_{Jq} - x_{Jq+r}| < \epsilon/2$. Therefore, $n \ge N$ implies

$$|x_n - \ell| = |x_{Jq+r} - \ell| \le |x_{Jq+r} - x_{Jq}| + |x_{Jq} - \ell| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof of the Lemma.

We have shown that some convergent series can be rearranged
so that the rearranged series converges to a different sum!

This is actually an example illustrating that if a series converges and the series of its absolute values diverges, then given any real number, or one of the two infinities, the series can be rearranged so that the new series, *that uses exactly the same terms,* converges to the given real number, or diverges to the chosen infinity. We can even arrange for it to diverge by oscillation. However, if the series of absolute values converges, then no matter how we rearrange the series, the new series converges to the same sum as the original one did.

A different approach, using power series, to find the sum of the original series

We know that if $|x| < 1$ the Geometric Series $\displaystyle\sum_{n=0}^{\infty} x^n$ converges to $\displaystyle\frac{1}{1-x}$. Therefore if $|x| < 1$ the series $\displaystyle\sum_{n=0}^{\infty}(-x)^n$ converges to $\displaystyle\frac{1}{1+x}$. We also know from Calculus that $\displaystyle\int_0^1 \frac{dx}{1+x} = \ln 2$. If we substitute the series into the integral, and interchange sum and integral (a step that is not always valid!) we get

$$\ln 2 = \int_0^1 \frac{dx}{1+x} = \int_0^1 \sum_{n=0}^{\infty}(-x)^n \, dx = (?) \sum_{n=0}^{\infty} \int_0^1 (-x)^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

We will prove that the equality is true, by using limits and the formula that we have for the partial sums of the Geometric Series. We write (when $|x| < 1$)

$$s_n(x) := \sum_{k=0}^{n} (-x)^n = \frac{1-(-x)^{n+1}}{1-(-x)} = \frac{1-(-x)^{n+1}}{1+x} = \frac{1}{1+x} - \frac{(-x)^{n+1}}{1+x}.$$

Hence

$$\int_0^1 \frac{1}{1+x}\,dx - \int_0^1 s_n(x)\,dx = \int_0^1 \frac{(-x)^{n+1}}{1+x}dx,$$

so that

$$\left|\int_0^1 \frac{1}{1+x}\,dx - \int_0^1 s_n(x)\,dx\right| \leq \int_0^1 \frac{x^{n+1}}{1+x}dx < \int_0^1 x^{n+1}\,dx = \frac{1}{n+1}.$$

In other words, $\int_0^1 s_n(x)\,dx \to \ln 2$ as $n \to \infty$. But $\int_0^1 s_n(x)\,dx = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}$, so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \ln 2.$$

This worked because we knew exactly what the remainders are in Taylor's Formula, and we were able to estimate them! We do not often have that luxury!