Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [9] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a and that for all $x \in \mathbb{R}$, $|f(x)| \le M$, where $M > 0$. Give an ϵ–δ proof that $g(x) := f(x)^2$ is continuous at a.

$$|g(x) - g(a)| = |f(x) - f(a)||f(x) - f(a)| \le |f(x) - f(a)|(|f(x)| + |f(a)|) \le |f(x) - f(a)|(2M).$$

Let $\epsilon' := \epsilon/2M$. Since f is continuous at a there exists $\delta' > 0$ such that $|x - a| < \delta' \Rightarrow |f(x) - f(a)| < \epsilon'$. Let $\delta := \delta'$. Then $|x - a| < \delta \Rightarrow |g(x) - g(a)| < 2M\epsilon' = \epsilon$.

(2) [12] Prove that $f(x) = x$ is continuous at a for all $a \in \mathbb{R}$, and use theorems and induction (starting at 1) to prove that x^n is continuous at a for all $a \in \mathbb{R}$, for all positive integers n. Identify or state theorems you use.

$|f(x) - f(a)| = |x - a|$, so given $\epsilon > 0$ let $\delta := \epsilon$. Thus $P(1)$, the statement "x^1 is continuous," is true. If $P(n)$, the statement "x^n is continuous," is true, then $x^{n+1} = x^n x$ is continuous by the Product Limit Theorem.

(3) [5] State the Well-Ordering Theorem.

Every non–empty set of natural numbers has a least element.

(4) [15] Define *function.* Let $X := \{0, 1, 2, 3\}$ and $Y := \{4, 5, 6\}$. Here are three subsets of $X \times Y$:
$f := \{(2, 6), (1, 5), (0, 5), (2, 4)\}$, $g := \{(2, 4), (0, 6), (3, 6), (1, 5)\}$, $h := \{(0, 4), (1, 5), (2, 6)\}$.
One of these is a function __ $: X \to Y$ and the other two are not. Which one is, and why are the other two not?

If X and Y are non-empty sets, a subset $f \subseteq X \times Y$ is a *function* if
(i) for every $x \in X$ there exists $y \in Y$ such that $(x, y) \in f$;
(ii) for every $x \in X$ and for every $y_1 \in Y$, $y_2 \in Y$, if $(x, y_1) \in f$ and $(x, y_2) \in f$ then $y_1 = y_2$.

f is not a function because $(2, 4)$ and $(2, 6)$ are in f and $4 \ne 6$.
g is a function because (i) and (ii) hold for g.
h is not a function because $x = 3$ is not the first member of any ordered pair in h.

(5) [9] Convert to a statement "in logic," with quantifiers, and prove axiomatically:
Whenever a, b and c are in a field F and $a + c = b + c$ then $a = b$.

$$(\forall a \in F)(\forall b \in F)(\forall c \in F)(a + c = b + c \Rightarrow a = b).$$

Proof:

$a + c = b + c$	(given)
$(a + c) + (-c) = (b + c) + (-c)$	(Add. inverses, Addition, Subst.)
$a + (c + (-c)) = b + (c + (-c))$	(Assoc.)
$a + 0 = b + 0$	(Subst.)
$a = b$	(Add. identity).