Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [11] Use the *Completeness Axiom* to prove that $\mathbb{N}$ is not bounded above.

Suppose instead that $\mathbb{N}$ is bounded above. Then since $0 \in \mathbb{N}$, $\mathbb{N}$ is non-empty and bounded above, so $\sigma := \sup \mathbb{N}$ exists in $\mathbb{R}$. Since $\sigma - 1 < \sigma$, $\sigma - 1$ cannot be an upper bound for $\mathbb{N}$, so there exists $m \in \mathbb{N}$ such that $m > \sigma - 1$. But then $m + 1 \in \mathbb{N}$ since $\mathbb{N}$ is inductive. However, $m + 1 > \sigma$, which contradicts the fact that σ is an upper bound for $\mathbb{N}$. This contradiction means that $\mathbb{N}$ was not bounded, after all.

(2) [11] Use (and cite!) axioms to prove that for all real numbers x and y, $-(x - y) = y - x$.

We know that there exists $w \in \mathbb{R}$ such that $(x - y) + w = 0$ (Axiom on existence of add. inverses).
By a Theorem we proved, additive inverses are unique.
We define $x - y$ by $x + (-y)$, where $-y$ is the unique additive inverse of y.
We show now that $(x - y) + (y - x) = 0$, which by uniqueness shows $y - x = w = -(x - y)$.:

$$\begin{aligned}
(x-y)+(y-x) &= (x+(-y))+(y-x) && \text{Definition and Subst.}\\
&= x+((-y)+(y-x)) && \text{Assoc.}\\
&= x+((-y)+(y+(-x))) && \text{Definition and Subst.}\\
&= x+(((-y)+y)+(-x)) && \text{Assoc.}\\
&= x+((y+(-y))+(-x)) && \text{Commut.}\\
&= x+(0+(-x)) && \text{Add. Inv. and Subst.}\\
&= x+(-x) && \text{Add. Ident. and Subst.}\\
&= 0 && \text{Add. Inv. and Subst.}\\
&\Rightarrow y-x = -(x-y) && \text{Uniqueness of Add. Inv.}
\end{aligned}$$

(3) [11] Let $f(x) = \dfrac{x^2 - 9}{x^3 - 27}$ if $x \neq 3$. Define $f(3)$ so that f is continuous at 3, *if possible.* Justify your answer in some, but not excessive, detail!

$$\text{If } x \neq 3, \quad f(x) = \frac{x^2-9}{x^3-27} = \frac{(x-3)(x+3)}{(x-3)(x^2+x3+3^2)} = \frac{x+3}{x^2+3x+9} =: g(x).$$

The denominator is never zero, and is continuous; the numerator is continuous. Thus $g(x)$ is continuous so

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x+3}{x^2+3x+9} = \lim_{x\to 3} g(x) = g(3) = \frac{3+3}{3^2+3\cdot 3+9} = 2/9.$$

If we now define $f(3) = 2/9$, f is continuous at 3.

(4) [11] *Given* that $f(x) = x$ is continuous at a for all $a \in \mathbb{R}$, use theorems and induction (starting at 1) to prove that x^{-n} is continuous at a for all $a \in \mathbb{R} \setminus \{0\}$, for all positive integers n. Identify or state theorems you use.

When $a \neq 0$, $1/x$ is continuous at a. This is our $P(1)$. If $P(n)$ is true, then $1/x^{n+1} = (1/x^n)(1/x)$ is the product of two functions continuous at $a \neq 0$, hence is continuous at $a \neq 0$.

(5) [6] State the Squeeze, or Sandwich, Theorem.

If $f(x) \to L$ and $h(x) \to L$ as $x \to a$ and $f(x) \leq g(x) \leq h(x)$ near a, then $g(x) \to L$ as $x \to a$.