Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [11] State the *Intermediate Value Theorem.* Prove that there exists $x \in \mathbb{R}$ such that $x^3 = 10$.

If f is continuous on the closed and bounded interval $[a, b]$ and $f(a) < y < f(b)$ for some $y \in \mathbb{R}$, then there exists $c \in [a, b]$ such that $f(c) = y$.

Let $a = 0$, $b = 3$. Then with $f(x) := x^3$ we have $f(0) = 0 < 10 < 27 = f(3)$ so by the IVT $\sqrt[3]{10}$ exists.

(2) [11] Prove that, if a real sequence $\{x_n\}$ is bounded above and increasing then it has a limit. How is the limit related to the sequence?

We let $S := \{x_n : n \in \mathbb{N}\}$. S is given to be bounded above and $S \neq \emptyset$ since $x_0 \in S$. Thus $L := \sup S$ exists in $\mathbb{R}$. We show that $x_n \to L$. Given $\epsilon > 0$ there exists $y \in S$ such that $y > L - \epsilon$. But $y = x_N$ for some $N \in \mathbb{N}$. If $n > N$ we have $0 \le L - x_n \le L - x_N < \epsilon$, since L is an upper bound and since the sequence is increasing. Thus $x_n \to L$. The limit is the supremum of the set of numbers in the sequence.

(3) [11] State the Extreme Value Theorem. Suppose that $f : (0, \infty) \to \mathbb{R}$ is continuous on $(0, \infty)$ and that $f(x) \to +\infty$ as $x \to 0$ and as $x \to \infty$. An example: $f(x) := x + \frac{1}{x}$. Prove that there exists $x_m \in (0, \infty)$ such that $f(x_m) \le f(x)$ for all $x \in (0, \infty)$. Hints: Sketch a graph; plot $(1, f(1))$ on your graph. Why are there points $0<a<b<\infty$ such that $f(x)>f(1)$ if $0<x<a$ or $b<x<\infty$? Use the Extreme Value Theorem now.

If f is continuous on a bounded interval, then f assumes maximum and minimum values there.

Since $f(x) \to +\infty$ as $x \to 0$ and as $x \to \infty$, there exists $0 < a < 1$ and $1 < b < \infty$ such that $f(x) > f(1)$ for all $0 < x \le a$ and for all $x \ge b$. On the interval $[a, b]$, $f(x)$ is continuous, so in particular there is $x_m \in [a, b]$ such that $f(x_m) \le f(x)$ for all $x \in [a, b]$. Since $f(x_m) \le f(1)$ and $f(x) > f(1)$ in $(0, a)$ and in (b, ∞), we have $f(x_m) \le f(x)$ for all $x > 0$.

(4) [11] Assume you have proved that for all x in any field F, $(-1)x = -x$. Now prove axiomatically that, if x is in an ordered field $\mathcal{O}$ and $x \neq 0$, then $x^2 > 0$.

- If $x > 0$ then $x^2 > 0$ by one of the order axioms.

- If $x \neq 0$ and x is not positive then $-x$ is positive, by Trichotomy. Thus $(-x)^2 > 0$. But $(-x)^2 = (-x)(-x) = (-1)x(-1)x = (-1)(-1)x^2$. We have proved that $-(-x) = x$. By this and the given theorem $(-1)(-1) = -(-1) = 1$. Hence $0 < (-x)^2 = x^2$.

(5) [6] State the Bolzano-Weierstrass Theorem.

Every bounded sequence of real numbers has a convergent subsequence.