Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [11] State the *Cauchy Criterion.* Suppose that $\{x_n\}$ is an increasing sequence that has a Cauchy subsequence. Prove that $\{x_n\}$ is a Cauchy sequence.

A sequence is a Cauchy sequence if and only if it converges.

Since a subsequence $\{x_{n_k}\}$ is Cauchy, it converges, say to L. But $\{x_{n_k}\}$ is also an increasing sequence, so $x_{n_k} \le L$. Then for each k, since $k \le n_k$, we have $x_k \le x_{n_k} \le L$. Thus $\{x_n\}$ is increasing and bounded above, so $\{x_n\}$ converges, hence is a Cauchy sequence.

(2) [11] State the *Schwarz Inequality* for $\mathbb{R}^n$ and use it to prove the Triangle Inequality.

For all vectors x and y in $\mathbb{R}^n$, $|x \bullet y| \le |x||y|$, with equality if and only if one of x and y is a multiple of the other.

$$|x+y|^2 = |x|^2 + 2x \bullet y + |y|^2 \le |x|^2 + 2|x \bullet y| + |y|^2 \le |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2.$$

(3) [11] Suppose the sequence $\{x_n\}$ converges to L. Show that if $\mathcal{F}$ is a family of open intervals that covers $\{x_n\}$ and $\{L\}$, then finitely many of the open intervals in $\mathcal{F}$ suffice to cover $\{x_n\}$ and $\{L\}$.

Since $\mathcal{F}$ covers $\{L\}$, there is an open interval $I := (A, B) \in \mathcal{F}$ such that $L \in (A, B)$.
We let $\epsilon := \min\{L-A, B-L\}$, so that $(L-\epsilon, L+\epsilon) \subseteq (A, B)$. Since $\{x_n\}$ converges to L there exists $N \in \mathbb{N}$ such that $n \ge N \Rightarrow |x_n - L| < \epsilon$. That is, $n \ge N \Rightarrow x_n \in (L-\epsilon, L+\epsilon) \subseteq (A, B)$, so that $n \ge N \Rightarrow x_n \in (A, B) \in \mathcal{F}$. If $0 \le n < N$, there exists $I_n := (A_n, B_n) \in \mathcal{F}$ such that $x_n \in (A_n, B_n)$, so the intervals $\{I, I_0, \ldots, I_{N-1}\}$ suffice to cover $\{x_n\}$ and $\{L\}$.

(4) [11] Assume you have proved that for all x in any field F, $0 \cdot x = 0$. Prove axiomatically that for all x in any field F, $(-1)x = -x$.

We want to show that $x + (-1)x = 0$. Since additive inverses are unique (Axiom or Theorem), this would show that $(-1)x = -x$. Now $x + (-1)x = 1 \cdot x + (-1)x$ (mult. ident. & subst) $= (1 + (-1))x$ (distrib). Thus $x + (-1)x = (1 + (-1))x = 0 \cdot x = 0$ (subst & given).

(5) [6] State the Heine-Borel Theorem.

If a family of balls covers a closed rectangle in $\mathbb{R}^n$ then a finite subfamily suffices to cover it (the one-dimensional version is OK too).