

Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [11] Define  $F(x, y) := \frac{x^2 y^2}{x^2 + y^2}$  if  $(x, y) \neq (0, 0)$  and define  $F(0, 0) := 0$ . Calculate  $\frac{\partial F}{\partial x}$  and  $\frac{\partial F}{\partial y}$  for every  $(x, y) \in \mathbb{R}^2$ . Prove or disprove that  $F$  is differentiable at  $(0, 0)$ .

$$\text{If } (x, y) \neq (0, 0) \text{ then } \frac{\partial F}{\partial x} = \frac{2xy^2}{r^2} - 2\frac{x^2 y^2}{r^3} \frac{x}{r} = \frac{2xy^2 r^2 - 2x^3 y^2}{r^4} = \frac{2xy^4}{r^4} \text{ and, similarly, } \frac{\partial F}{\partial y} = \frac{2x^4 y}{r^4}.$$

If  $(x, y) = (0, 0)$  then  $F$  is identically zero on each coordinate axis, so  $\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$  when  $(x, y) = (0, 0)$ . Thus

$$\frac{|\text{Rem}|}{r} = \frac{x^2 y^2}{r^3} \leq \frac{r^2 r^2}{r^3} = r \rightarrow 0 \text{ as } r \rightarrow 0, \text{ so } F \text{ is differentiable at } (0, 0).$$

(2) [11] Prove that if  $f : (a, b) \rightarrow \mathbb{R}$  and if  $f'(x) > 0$  for every  $x \in (a, b)$  then  $f$  is strictly increasing in  $(a, b)$ .

Suppose that  $a < x < y < b$ . We have to show:  $f(x) < f(y)$ . By the MVT,  $f(y) - f(x) = f'(c)(y - x)$  for some  $c$  strictly between  $x$  and  $y$ , and  $f'(c) > 0$  so  $f(y) - f(x) = \text{positive} \cdot \text{positive} > 0$ .

(3) [11] Prove that the function  $F(x) := x + \sin x$  takes on all real values.

We use the IVT: Given  $y \in \mathbb{R}$  we let  $a = y - 2$ ,  $b = y + 2$ . Then  $F(a) = a + \sin a = y - 2 + \sin a \leq y - 2 + 1 = y - 1 < y$ . And  $F(b) = b + \sin b = y + 2 + \sin b \geq y + 2 - 1 = y + 1 > y$ . Since  $F$  is everywhere continuous the IVT is applicable on  $[a, b]$  so there exists  $c$ ,  $a < c < b$ , such that  $F(c) = y$ .

(4) [11] Prove axiomatically that for all  $x$  in any field  $F$ ,  $0 \cdot x = 0$ .

$$\begin{aligned} 0 \cdot x &= (0 + 0) \cdot x \text{ Add. Ident, Subst.} \\ &= 0 \cdot x + 0 \cdot x \text{ Mult Comm., Distrib., Mult Comm.} \end{aligned}$$

We let  $w := -(0 \cdot x)$ . Then since (as we have shown)

$$\begin{aligned} 0 \cdot x &= 0 \cdot x + 0 \cdot x, \\ 0 \cdot x + w &= 0 \cdot x + (0 \cdot x + w) \text{ Addition, Subst., Add. Assoc., so that} \\ 0 &= 0 \cdot x + 0 = 0 \cdot x \text{ Add. Inv., Subst., Add. Ident., as desired.} \end{aligned}$$

(5) [6] State the Generalized Mean Value Theorem.

Suppose that  $f$  and  $F$  are differentiable on  $(a, b)$  and continuous on  $[a, b]$  and  $F'(t) \neq 0$  for any  $t \in (a, b)$ . Then there exists  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)}.$$