

Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 50 points.

(1) [11] Define $F(x, y) := \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$ if $(x, y) \neq (0, 0)$ and define $F(0, 0) := 0$. Prove that F is not differentiable at $(0, 0)$ and prove that F^2 is differentiable at $(0, 0)$.

(i) Find partials of F at $(0, 0)$: $\frac{\partial F}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{F(h, 0) - F(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^2}{\sqrt{h^2}h} = \lim_{h \rightarrow 0} \frac{h}{|h|}$ does not exist, so the partials do not exist at $(0, 0)$ so F is not differentiable at $(0, 0)$.

(ii) Find partials of F^2 at $(0, 0)$: $\frac{\partial F^2}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{F^2(h, 0) - F^2(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^4}{h^2h} = \lim_{h \rightarrow 0} h = 0$. Similarly, $\frac{\partial F^2}{\partial y}(0, 0) = 0$. Thus $F^2 - F(0) - \nabla F(0) = F^2$.

(iii) Show $\frac{|\text{Rem}|}{r} \rightarrow 0$. For F^2 , at $(0, 0)$, $\text{Rem} = F^2$, so $\frac{|\text{Rem}|}{r} = \frac{(x^2 - y^2)^2}{r^2 r} \leq \frac{(x^2 + y^2)^2}{r^2 r} = \frac{r^4}{r^3} = r \rightarrow 0$, which proves that F^2 is differentiable at $(0, 0)$.

(2) [11] Suppose that $f(x)$ is differentiable in (a, b) and that $f'(x)$ is differentiable at $c \in (a, b)$. Prove that $\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$. Hint: Use GMVT, then def. of deriv. What's the variable here?

The variable here is h . If $\delta > 0$ is so small that $a < c - \delta < c + \delta < b$, and $|h| < \delta$, our interval for GMVT is $I = [c - |h|, c]$ if $h < 0$ and $I = [c, c + h]$ if $h > 0$. The roles of f and F in the GMVT are played by $f(c+h) - 2f(c) + f(c-h)$ and h^2 , respectively. By GMVT there exists $h' \in I$ such that

$$\frac{(f(c+h) - 2f(c) + f(c-h)) - (f(c+0) - 2f(c) + f(c-0))}{h^2 - 0^2} = \frac{f'(c+h') - f'(c-h')}{2h'}$$

Then $\frac{f'(c+h') - f'(c-h')}{2h'} = \frac{f'(c+h') - f'(c)}{2h'} + \frac{f'(c) - f'(c-h')}{2h'} = \frac{f'(c+h') - f'(c)}{2h'} + \frac{f'(c-h') - f'(c)}{2(-h')}$, which tends to $f''(c)$ as $h \rightarrow 0$ (since h' is strictly between 0 and h , when $h \rightarrow 0$ then $h' \rightarrow 0$, by the Squeeze Theorem).

(3) [11] Prove that the set of all sequences of zeroes and ones is uncountable.

Suppose not. Then every sequence of zeroes and ones can be put into a list: a sequence of sequences of zeroes and ones: $\{s_n\}$, where each s_n is a sequence of zeroes and ones. Let s_{nj} denote the zero or one of s_n at position j . Let $\eta_n := 1 - s_{nn}$. Then the sequence η is one of the s_n , say $\eta = s_N$. But $\eta_N = 1 - s_{NN} \neq s_{NN}$, so $\eta \neq s_N$. This (" $\eta = s_N$ and $\eta \neq s_N$ ") is a contradiction, so it is false that every sequence of zeroes and ones can be put into a list indexed by \mathbb{N} . Hence the set of all sequences of zeroes and ones is uncountable.

(4) [11] Prove that $x = 0$ if and only if for all $\epsilon > 0$, $|x| < \epsilon$.

If $x = 0$, then $|x| = 0 < \epsilon$ for every positive ϵ . If, on the other hand, for all $\epsilon > 0$, $|x| < \epsilon$, we suppose, on the contrary, that $x \neq 0$. Thus $|x| > 0$. Let us put $\epsilon := |x|/2$. Then $|x| < \epsilon = |x|/2$ is false, contradicting our hypothesis that for all $\epsilon > 0$, $|x| < \epsilon$.

(5) [6] Define Riemann integrable function on a "box" in \mathbb{R}^n .

If f is defined and bounded on a "box" in \mathbb{R}^n then f is Riemann integrable on the "box" if there exists a number RI such that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|\sum_{\Delta_i \in F} F(\xi_i)V(\Delta_i) - RI| < \epsilon$ whenever F is a subdivision or partition of the box into sub-boxes Δ_i , the longest diameter of each Δ_i , being less than δ (or one can say $\text{mesh}(F) < \delta$).