

**Quiz 7 Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.**

(1) [15] State and prove the Cauchy Mean Value Theorem.

CMVT: If  $f(x)$  and  $g(x)$  are continuous on  $[a, b]$ , differentiable on  $(a, b)$  and  $g'(x) \neq 0$  on  $(a, b)$ , then  $g(b) - g(a) \neq 0$  and there exists  $c \in (a, b)$  such that  $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ .

Proof: Let  $h(x) := (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a))$ . Then  $h(a) = 0 = h(b)$ . Also,  $h(x)$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ . The Mean Value Theorem applies, so there exists  $c \in (a, b)$  such that  $0 = h(b) - h(a) = h'(c) = f'(c)(g(b) - g(a)) - (f(b) - f(a))g'(c)$ . Thus  $\frac{f'(c)}{g'(c)}(g(b) - g(a)) = f(b) - f(a)$ . By MVT again,  $g(b) - g(a) = g'(d)(b - a) \neq 0$  for some  $d \in (a, b)$ , so  $g(b) - g(a) \neq 0$ . Thus  $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$ .

(2) [15] State Taylor's Theorem, with Lagrange Remainder. Find the Taylor Formula of order four of  $\sqrt{x}$  about the "base point"  $x_o = 4$ . Be sure to include the remainder. For 5 extra points, find a formula for all the coefficients.

Taylor's Theorem: If  $f(x)$  is continuous on  $(a, b)$  and for  $1 \leq n \leq N + 1$ ,  $f^{(n)}(x)$  exists on  $(a, b)$  then for each  $x_o \in (a, b)$  and each  $x \in (a, b)$ , there exists  $c_{N+1} \in (a, b)$ , strictly between  $x$  and  $x_o$ , such that

$$f(x) = \sum_{n=0}^N \frac{f^{(n)}(x_o)}{n!} (x - x_o)^n + \frac{f^{(N+1)}(c_{N+1})}{(N + 1)!} (x - x_o)^{N+1}.$$

For  $f(x) = \sqrt{x}$ ,  $f'(x) = \frac{1}{2}x^{-1/2}$ ,  $f''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-3/2} = \frac{-1}{4} x^{-3/2}$ ,  $f'''(x) = \frac{-1}{4} \cdot \frac{-3}{2} x^{-5/2} = \frac{3}{8} x^{-5/2}$ ,  $f^{(4)}(x) = \frac{3}{8} \cdot \frac{-5}{2} x^{-7/2} = \frac{-15}{16} x^{-7/2}$ ,  $f^{(5)}(x) = \frac{-15}{16} \cdot \frac{-7}{2} x^{-9/2} = \frac{105}{32} x^{-9/2}$ .

When  $x_o = 4$ , and for  $0 \leq k \leq 4$ ,  $\frac{f^{(k)}(4)}{k!} = 2, 1/4, -1/4 \cdot 8 \cdot 2!, 3/8 \cdot 32 \cdot 3!, -15/16 \cdot 128 \cdot 4!$  so that  $\sqrt{x} = 2 + \frac{x-4}{2} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{16 \cdot 128 \cdot 8} + \frac{105(x-4)^5}{32 \cdot 5!} c^{-9/2} = 2 + \frac{x-4}{2} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} - \frac{5(x-4)^4}{2^{14}} + \frac{7(x-4)^5}{256} c^{-9/2}$ , where  $c$  is strictly between 4 and  $x$ .

[Here, for arbitrary  $k \geq 2$ ,  $\frac{f^{(k)}(4)}{k!} = (-1)^{k+1} \frac{1 \cdot 3 \cdots (2k-3)}{2^k k! 2^{2k-1}} = (-1)^{k+1} \frac{1 \cdot 3 \cdots (2k-3)}{2^{3k-1} k!}$  ]

(3) [12] Prove that the Harmonic Series diverges.

Harmonic Series:  $\sum_{n=1}^{\infty} \frac{1}{n}$ . We have, dividing the terms into blocks between powers of 2,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}\right) + \cdots$$

Since the block  $\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}$  has  $2^n$  terms, each (except the last)  $> \frac{1}{2^{n+1}}$ , the sum over the block is at least  $> \frac{2^n}{2^{n+1}} > 1/2$ . Therefore the partial sum  $S_{2^{n+1}} > \frac{n+3}{2}$  (because of the initial 1), and hence the (increasing) sequence of partial sums diverges to  $+\infty$ .

(4) [18] Test these series for convergence. Justify your answers! Some answers may depend on  $x$ . If  $x$  appears, assume  $x > 0$ .

(a)  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ ; (b)  $\sum_{n=0}^{\infty} \frac{3n+1}{n^3+1}$ ; (c)  $\sum_{n=0}^{\infty} \frac{n+2}{(n+1)(n+3)}$ ; (d)  $\sum_{n=0}^{\infty} \frac{n!}{n^n}$ ; (e)  $\sum_{n=0}^{\infty} \frac{n^2+1}{2^n}$ ; (f)  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

(a) Ratio Test; ratio is  $x/(n+1) \rightarrow 0$  so the series converges for all  $x > 0$ .

(b) Comparison Test:  $\frac{3n+1}{n^3+1} < \frac{4n}{n^3} = 4/n^2$  (in the numerator,  $1 \leq n$ , and the denominator is larger than  $n^2$ ) and we know  $\sum 1/n^2$  converges, so series (b) converges.

(c) Limit Comparison Test:  $\frac{n+2}{(n+1)(n+3)} \div \frac{1}{n} = \frac{n(n+2)}{(n+1)(n+3)} \rightarrow 1$  and the Harmonic Series diverges, so series (c) diverges.

(d) Ratio Test, plus a fact given during the Quiz, namely  $(1 + \frac{1}{n})^n \rightarrow E > 1$ , and finite: the ratio is  $\frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = (\frac{n}{n+1})^n = 1/(\frac{n+1}{n})^n = 1/(1 + \frac{1}{n})^n \rightarrow 1/E < 1$ . Thus series (d) converges.

(e) Ratio Test: ratio is  $\frac{(n+1)^2+1}{n^2+1} \frac{2^n}{2^{n+1}} = \frac{(n+1)^2+1}{n^2+1} \frac{1}{2} \rightarrow 1/2 < 1$ , so series (e) converges.

(f) Root Test, or Ratio Test or Geometric Series! Geometric Series:  $x^n/2^n = (x/2)^n$  so the series converges if  $0 < x < 2$ , diverges if  $x > 2$ , and diverges when  $x = 2$  because the terms do not tend to zero.

**Quiz 6 Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.**

(1) [15] State the Mean Value Theorem. Suppose that  $f$  and  $g$  are defined on  $\mathbb{R}$ , are differentiable everywhere, and:  $f'(x) \equiv g'(x)$  and there exists  $x_o$  such that  $f(x_o) = g(x_o)$ . Prove that  $f(x) \equiv g(x)$ .

MVT: If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then there exists  $c \in (a, b)$  such that  $f(b) - f(a) = f'(c)(b - a)$ .

Let  $h(x) := f(x) - g(x)$ . By hypothesis,  $h'(x) \equiv 0$  and  $h(x_o) = 0$ . By MVT there exists  $c$  strictly between  $x$  and  $x_o$  such that  $h(x) = h(x) - h(x_o) = h'(c)(x - x_o) = 0$ . Thus  $f(x) \equiv g(x)$ .

(2) [15] Prove that  $\sqrt{y}$  is strictly increasing on  $[0, \infty)$ .

Ans #1: By a Theorem proved in class, a function is strictly increasing if  $f'(x) > 0$  for all  $x$ . Then  $(\sqrt{y})' = 1/2\sqrt{y} > 0$  so  $\sqrt{y}$  is strictly increasing on  $[0, \infty)$ .

Ans #2: Let  $f(y) := \sqrt{y}$ . By MVT, if  $0 \leq a < b < +\infty$ , there exists  $c$  strictly between  $a$  and  $b$  such that  $f(b) - f(a) = f'(c)(b - a) = (b - a)/2\sqrt{c} > 0$ .

(3) [15] Suppose that  $F : (a, b) \rightarrow \mathbb{R}$  is one-to-one and continuous. Prove that  $F((a, b))$  is an open interval. Identify any Theorems that you use!

A Theorem proved in class (and in the text) says that if a function is continuous on an interval then the image of the function is an interval. Thus  $F((a, b))$  is an interval. To show the interval is open, let  $y \in F((a, b))$ . In class we proved that  $F$  is strictly monotone. Let us assume that  $F$  is strictly increasing. Then  $y = F(x)$ , where  $x \in (a, b)$ . Thus there exists  $\delta > 0$  such that  $(x - \delta, x + \delta) \subseteq (a, b)$ . Therefore  $F(x + \delta/2) > F(x) = y > F(x - \delta/2)$ . By the IVT, every  $z$  in  $(F(x - \delta/2), F(x + \delta/2))$  is a value of  $F$ . Hence  $(F(x - \delta/2), F(x + \delta/2)) \subseteq F((a, b))$ .

(4) [15] State the Intermediate Value Theorem for Derivatives. Prove that if a function differentiable on  $(a, b)$  and continuous on  $[a, b]$  has an interior maximum or minimum at  $c \in (a, b)$  then  $f'(c) = 0$ . This was used in the proof of IVT'.

IVT': If  $f$  is differentiable on an open interval  $(a, b)$  then for every  $a < x_1 < x_2 < b$ , if  $y$  is strictly between  $f'(x_1)$  and  $f'(x_2)$  then there exists  $c \in (x_1, x_2)$  such that  $f'(c) = y$ .

If  $a < c < b$  and  $f$  has a maximum at  $c$  then  $\frac{f(c) - f(x)}{c - x} \geq 0$  if  $a < x < c$  and  $\frac{f(c) - f(x)}{c - x} \leq 0$  if  $c < x < b$ . Thus the limit from the left (of the difference quotient) is at least zero, while the limit from the right of the difference quotient is, at most, zero. Since differentiability implies that the two one-sided limits are equal, the limit (the derivative!) must be zero.

**Quiz 5 Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.**

**Note:** On Quiz 4 you showed that there exists a function "sqrt," defined on  $(0, \infty)$ , such that  $(\text{sqrt}(y))^2 = y$  for all  $y > 0$ . You may use the notation  $\sqrt{y} = \text{sqrt}(y)$ .

(1) [15] Define  $\lim_{x \rightarrow x_o^+} f(x)$ . Show that, if we define  $\text{sqrt}(0) = 0$ , then  $\lim_{y \rightarrow 0^+} \text{sqrt}(y) = \text{sqrt}(0)$ .

Definition:  $\lim_{x \rightarrow x_o^+} f(x)$  means there exists  $L \in \mathbb{R}$  such that for all  $\epsilon > 0$  there exists  $\delta > 0$  such that for all  $x$  in the domain of  $f$ ,  $x_o < x < x_o + \delta \Rightarrow |f(x) - L| < \epsilon$ .

Given  $\epsilon > 0$ , we use  $L = 0$  and want  $|\sqrt{y} - 0| < \epsilon$  if  $0 < y < \delta$ . As in #2,  $\sqrt{y} < \epsilon$  if and only if  $y < \epsilon^2$ . Thus we set  $\delta = \epsilon^2$ .

(2) [15] Prove this inequality: For all positive numbers  $a$  and  $b$ ,  $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ . You may use without proof: "For  $x > 0$  and  $y > 0$ ,  $x > y$  if and only if  $x^2 > y^2$ ."

$$\sqrt{a+b} < \sqrt{a} + \sqrt{b} \iff a + b < (\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{a} \cdot \sqrt{b} + b.$$

Since  $2\sqrt{a} \cdot \sqrt{b} > 0$  the inequality on the right is true. Thus  $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ .

(3) [15] Define *differentiable at a point*. Use the definition to find  $\frac{d}{dy}\sqrt{y}$  when  $y > 0$ . Why is it therefore true that  $\sqrt{y}$  is continuous on  $[0, \infty)$  (you'll need # 1 at one point)?

Definition:  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$  exists.

$$\frac{\sqrt{y} - \sqrt{y_0}}{y - y_0} = \frac{\sqrt{y} - \sqrt{y_0}}{y - y_0} \frac{\sqrt{y} + \sqrt{y_0}}{\sqrt{y} + \sqrt{y_0}} = \frac{y - y_0}{y - y_0} \frac{1}{\sqrt{y} + \sqrt{y_0}} = \frac{1}{\sqrt{y} + \sqrt{y_0}} \rightarrow \frac{1}{2\sqrt{y_0}}.$$

By # 1,  $\sqrt{y}$  is continuous at  $y_0 = 0$ . If  $y_0 > 0$  then  $\sqrt{y}$  is differentiable at  $y_0$ , hence continuous at  $y_0$ .

(4) [15] State the Chain Rule. Given that  $a \neq 0$ , use the Chain Rule to find the derivative of  $r(x) := \sqrt{a^2 + x^2}$ . Be sure you specify how you use the Chain Rule.

Chain Rule: If  $g$  is differentiable at  $x$  and  $f$  is differentiable at  $g(x)$  then  $f(g(x))$  is differentiable at  $x$  and  $f(g(x))' = f'(g(x))g'(x)$ .

Let  $f(y) = \sqrt{y}$ ,  $g(x) = a^2 + x^2$ . Since  $a \neq 0$ ,  $f$  is differentiable at  $g(x)$  and  $f'(g(x)) = 1/(2\sqrt{a^2 + x^2})$  and  $g'(x) = 2x$ . Thus  $f(g(x))' = 1/(2\sqrt{a^2 + x^2}) \cdot 2x = x/\sqrt{a^2 + x^2}$ . This formula is worth memorizing!!

**Quiz 4 Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.**

(1) [15] Suppose that  $f(x)$  is continuous at  $x_0$ . Give an epsilon-delta proof that  $f(x)^2$  is continuous at  $x_0$ . That is, "solve" for  $\delta$  in terms of  $\epsilon$ ,  $f$  and  $x_0$ .

To show: for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|\Delta x| = |x - x_0| < \delta \Rightarrow |f(x)^2 - f(x_0)^2| < \epsilon$ . Given  $\epsilon > 0$ ,

$$f(x)^2 - f(x_0)^2 = (f(x) - f(x_0))(f(x) + f(x_0)) = \Delta f (f(x_0 + \Delta x) + f(x_0)) = \Delta f (\Delta f + 2f(x_0)), \text{ so}$$

$$|f(x)^2 - f(x_0)^2| = |\Delta f| |\Delta f + 2f(x_0)| \leq |\Delta f| (|\Delta f| + 2|f(x_0)|). \text{ We want to replace } |\Delta f| \text{ by } 1.$$

We find  $\delta > 0$  such that  $|\Delta x| < \delta \Rightarrow |\Delta f| < \min\{1, \epsilon\}$ . We might have to "adjust" the  $\epsilon$ ! Now, if  $|\Delta x| < \delta$ , then  $|\Delta f| < \min\{1, \epsilon\}$  so

$$|f(x)^2 - f(x_0)^2| \leq |\Delta f| (|\Delta f| + 2|f(x_0)|) < \epsilon (|\Delta f| + 2|f(x_0)|) < \epsilon (1 + 2|f(x_0)|).$$

We wanted  $\epsilon$ , not  $\epsilon(1 + 2|f(x_0)|)$ , so we replace our original  $\epsilon$  by  $\epsilon/(1 + 2|f(x_0)|)$ , and this will give us a new value of  $\delta$ , smaller than the one we had. We'll call the new one by the same name.

(2) [15] State the Intermediate Value Theorem. Use it (and other things) to prove that for every positive  $y$ , there exists a unique positive  $x$  such that  $x^2 = y$ .

IVT: If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) < y < f(b)$  then there exists  $c \in (a, b)$  such that  $f(c) = y$ .

With  $f(x) = x^2$  and  $y > 0$  we need numbers  $a$  and  $b$  with  $f(a) < y$  and  $f(b) > y$ . We can choose  $a = 0$  and  $b = y + 1$ . Then  $f(a) = 0 < y$  and  $f(b) = y^2 + 2y + 1 > y$ , so by IVT there exists  $c$  in  $(a, b)$  such that  $f(c) = y$ . To show uniqueness, we suppose that  $x_1^2 = y = x_2^2$ , both positive. Then  $0 = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$ . Since  $x_1 + x_2 > 0$  we can cancel, so  $0 = x_1 - x_2$ . That is,  $x_1^2 = y = x_2^2 \Rightarrow x_1 = x_2$ .

(3) [15] Suppose that  $E$  is a closed set and that a sequence  $\{x_n\}$  of points of  $E$  converges to a number  $c$ . Prove that  $c \in E$ .

If  $\{x_n\}$  has a subsequence with distinct points,  $c$  is a limit point of  $E$  hence belongs to  $E$ . If no subsequence of  $\{x_n\}$  has distinct points, some term of the sequence is repeated infinitely often, and that value has to be arbitrarily close to  $c$  so it must be equal to  $c$ . But then  $c$  was already in the sequence, so  $c \in E$ .

(4) [15] Suppose that  $a_n \leq b_n$  for all  $n$ , and that  $b_n \rightarrow L$  as  $n \rightarrow \infty$ . Prove that  $\limsup a_n \leq L$ . If you prefer, you can prove that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that whenever  $n \in \mathbb{N}$  and  $n \geq N$ ,  $a_n < L + \epsilon$ .

Ans #1:  $\limsup_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} \sup_{n \geq m} a_n =: \lim_{m \rightarrow \infty} A_m$ . Since  $\sup_{n \geq m} a_n \leq \sup_{n \geq m} b_n$  and  $b_n \rightarrow L$ , we know that, given  $\epsilon > 0$  there exists  $N$  such that  $n \geq N \Rightarrow L - \epsilon < b_n < L + \epsilon$ . Hence, if  $m \geq N$ ,  $A_m \leq \sup_{n \geq m} b_n \leq L + \epsilon$ . Thus for every  $\epsilon > 0$ ,  $\limsup_{n \rightarrow \infty} a_n = \inf_{m} A_m \leq L + \epsilon$ . As  $\epsilon > 0$  is arbitrary,  $\limsup_{n \rightarrow \infty} a_n \leq L$ .

Ans #2: Suppose not. Then  $c \in E^c$ , an open set, so there exists  $\delta > 0$  such that  $(c - \delta, c + \delta) \subseteq E^c$ . But there exists  $N$  such that  $n \geq N \Rightarrow |x_n - c| < \delta$ , so  $x_N \in (c - \delta, c + \delta) \subseteq E^c$ . This is a contradiction;  $x_N \in E$ . Thus  $c \in E$ .

### Quiz 3

Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points. Drive safely!

(1) [10] Given a sequence  $\{x_n\}$  such that  $x_{2n}$  increases and  $x_{2n+1}$  decreases and  $x_{2n} \leq x_{2n+1}$  for all  $n$ . Prove that  $\{x_{2n+1} - x_{2n}\}$  converges. Hint: Draw.

Ans #1: We are given  $x_{2n} \leq x_{2n+1}$  and  $x_{2n+1} \leq x_{2n+2}$ . Thus  $\{x_{2n}\}$  increases (given) and is bounded above. Thus  $\{x_{2n}\}$  converges, say to  $\mathcal{E}$ . Also,  $x_{2n} \leq x_{2n+1} \leq x_{2n+2}$ , so  $\{x_{2n+1}\}$  is decreasing and bounded below. Thus  $\{x_{2n+1}\}$  converges, say to  $\mathcal{O}$ . Then by limit theorems,  $x_{2n+1} - x_{2n} \rightarrow \mathcal{E} - \mathcal{O}$ .

Ans #2:  $x_{2n+1} - x_{2n} = x_{2n+1} + (-x_{2n})$  and  $\{-x_{2n}\}$  is decreasing because  $\{x_{2n}\}$  is increasing. Thus  $x_{2n+1} - x_{2n}$  is the sum of two decreasing sequences, hence is decreasing. Also,  $x_{2n+1} - x_{2n} \geq 0$  so we get convergence.

(2) [16] Show that, if  $f(x)$  is continuous at  $c$ , and  $\{x_n\}$  is a sequence that converges to  $c$ , and  $f(x_n)$  is defined for all  $n$ , then  $f(x_n) \rightarrow f(c)$ . Hint: Wait before you use this hint! Use the  $\epsilon > 0$  you need for convergence to get a  $\delta > 0$  from continuity and then use that  $\delta$  as your “epsilon” to get the natural number  $N$  from the given convergence of  $\{x_n\}$  that you need for the convergence of  $\{f(x_n)\}$ .

We want to show: for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |f(x_n) - f(c)| < \epsilon$ .

We know  $f$  is continuous at  $c$  so we can find  $\delta > 0$  such that  $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$ .

We know  $x_n \rightarrow c$  so we can find  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |x_n - c| < \delta$ .

Thus  $n \geq N \Rightarrow |x_n - c| < \delta \Rightarrow |f(x_n) - f(c)| < \epsilon$ .

(3) [17] Define *open set*. Define *closed set*. Prove that, if a closed set is removed from an open set, then the set of points left behind is still an open set. Hint: DeMorgan!

A set  $S$  is open if for every  $x \in S$  there exists  $\delta > 0$  such that  $(x - \delta, x + \delta) \subseteq S$ . A set  $S$  is closed if  $S$  contains all its limit points.

If  $S$  is open and  $T \subseteq S$  is closed then the set of points left behind when  $T$  is removed from  $S$  is

$$\{x \in \mathbb{R} : x \in S \text{ and } x \notin T\} = \{x \in \mathbb{R} : x \in S \text{ and } x \in T^c\} = S \cap T^c.$$

Since  $S$  and  $T^c$  are both open so is their intersection.

(4) [17] Define *compact set*. Prove that if  $K$  is non-empty and compact, then  $\sup K$  and  $\inf K$  both exist, and both belong to  $K$ .

A set  $K$  is compact if every sequence  $\{x_n\}$  whose values are all in  $K$  has a subsequence that converges to an element of  $K$ .

If  $K$  is compact, it is closed and bounded. If non-empty then  $\sup K$  and  $\inf K$  both exist. To show  $\sup K \in K$ , suppose not. Then  $\sup K \in K^c$  which is an open set. Hence there exists  $\delta > 0$  such that

$(\sup K - \delta, \sup K + \delta) \subseteq K^c$ . This is a contradiction because there must exist an element of  $K$  greater than  $\sup K - \delta$ . The argument for  $\inf K \in K$  is similar.

**Quiz 2** Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [16] Show that  $S := \{3^n\}$  is an unbounded sequence.

Suppose, on the contrary, that  $\{3^n\}$  is a bounded sequence. Then there exists  $M \in \mathbb{R}$  such that  $3^n \leq M$  for all  $n \in \mathbb{N}$ . Since  $3^1 > 0$ ,  $M \geq 3 > 0$ . Therefore  $M/3 < M$ . There thus exists  $N \in \mathbb{N}$  such that  $M/3 < 3^N$ . But then  $M < 3^{N+1}$ , and  $3^{N+1}$  is a term in our sequence. This contradicts the fact that a supremum is an upper bound. Hence  $\{3^n\}$  is an unbounded sequence.

(2) [15] Define “strictly increasing sequence.” Prove that if  $\{n_k\}$  is a strictly increasing sequence of natural numbers then  $n_k \geq k$  for all  $k \in \mathbb{N}$ . Hint: Use induction.

A sequence  $\{x_n\}$  is strictly increasing if  $n < m \Rightarrow x_n < x_m$ . If  $\{n_k\}$  is a strictly increasing sequence of natural numbers then  $n_k \geq 0$  because 0 is the least element of  $\mathbb{N}$ .

Thus  $n_0 \geq 0$ .

If  $n_k \geq k$ , to show:  $n_{k+1} \geq k + 1$ . Since  $n_{k+1} > n_k$  and both are integers,  $n_{k+1} \geq n_k + 1 \geq k + 1$ , as desired.

Thus by induction,  $n_k \geq k$  for all  $k \in \mathbb{N}$ .

(3) [12] Let  $x_n := \frac{n}{n+2}$ ,  $n \geq 1$ . Find  $A := \sup_n x_n$ ,  $B := \inf_n x_n$  and  $L := \lim_{n \rightarrow \infty} x_n$ . Are any of  $A, B$  and  $L$  terms of the sequence  $\{x_n\}$ ? If so, which term? Be sure to explain your answers.

(4) [17] Show that every decreasing sequence of real numbers that is bounded below converges (i.e., has a limit).

**Quiz 1** Ask! Indicate your approach! Show your work! Good Luck! There are 2 pages, and 60 points.

(1) [14] State the *Completeness Axiom* [7]. Let  $S := \{x \in \mathbb{R} : x < 0\}$ . Verify that  $\sup S$  exists and show that  $\sup S = 0$  [7].

Completeness Axiom: If  $S$  is a non-empty subset of the real numbers that is bounded above, then a real number  $\sup S$  exists, that has the two properties

- (1)  $\sup S$  is an upper bound for  $S$ , and
- (2) If  $x$  is a real number and  $x < \sup S$ , then there exists  $s \in S$  such that  $s > x$ .

The given set  $S$  is non-empty since  $-1 \in S$ ;  $S$  is bounded above by 0. By the Completeness Axiom,  $\sup S$  exists.

To show  $\sup S = 0$ , we know 0 is an upper bound for  $S$ . Thus Property (1) holds. If  $x < 0$  then  $x < x/2 < 0$  and  $x/2 \in S$ , so Property (2) holds. This means that  $\sup S = 0$ .

(2) [15] Let  $X := \{1, 2\}$ . Write out  $X \times X$  (the Cartesian product of  $X$  with itself),  $2^X$  (the power set of  $X$ ), and construct a one-to-one correspondence between  $2^X$  and  $X \times X$ .

$$X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$2^X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

We define  $f : 2^X \rightarrow X \times X$  by:

$$f(\emptyset) = (1, 1), \quad f(\{1\}) = (1, 2), \quad f(\{2\}) = (2, 1) \quad \text{and} \quad f(\{1, 2\}) = (2, 2).$$

(3) [16] Suppose that  $A$  and  $B$  stand for Truth Values, and that  $a := 1$  if  $A$  is True, and  $a := 0$  if  $A$  is False, with similar definitions for  $B$ . The formula  $f(a, b) := ab$  is a numerical expression we can use to “model” the logic expression  $A \wedge B$ . Verify that this formula is correct by comparing its “inputs” and “outputs” to the Truth Table for  $A \wedge B$ . Note that  $f(a, b)$  is a function defined on the set  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . Find your own formula  $g(a, b)$  to “model”  $A \Rightarrow B$ . Verify that your formula is correct.

$A$	$B$	$A \wedge B$	$f(a, b) = a \cdot b$
$T$	$T$	$T$	$1 \cdot 1 = 1$
$T$	$F$	$F$	$1 \cdot 0 = 0$
$F$	$T$	$F$	$0 \cdot 1 = 0$
$F$	$F$	$F$	$0 \cdot 0 = 0$

Claim:  $g(a, b) = 1 - a(1 - b)$ , because “implies” is: NOT ( $A$  AND NOT  $B$ ), and NOT  $A$  has formula  $1 - a$ . The formula simplifies to  $1 - a + ab$ . Verification:

$A$	$B$	$A \Rightarrow B$	$g(a, b) = 1 - a + ab$
$T$	$T$	$T$	$1 - 1 + 1 \cdot 1 = 1$
$T$	$F$	$F$	$1 - 1 + 1 \cdot 0 = 0$
$F$	$T$	$T$	$1 - 0 + 0 \cdot 1 = 1$
$F$	$F$	$T$	$1 - 0 + 0 \cdot 0 = 1$

(4) [15] Prove that the equation  $x^2 = 3$  has no solutions  $x$  in the set of rational numbers.

Proof by contradiction: Suppose  $(m/n)^2 = 3$ , and that  $m$  and  $n$  have no common factors. Then  $m^2 = 3n^2$ .

LEMMA: If  $m^2$  is divisible by 3 then  $m$  is divisible by 3. Proof of LEMMA: prove the contrapositive: If  $m$  is not divisible by 3 then  $m^2$  is not divisible by 3. Now,  $m$  is not divisible by 3 means that  $m = 3q + r$ , where  $r = 1$  or  $r = 2$ . Then  $m^2 = 9q^2 + 6qr + r^2 = 3(3q^2 + 2qr) + r^2$ . If  $r = 1$ ,  $m^2 = 3Q + 1$ , where  $Q = 3q^2 + 2qr$ . If  $r = 2$ ,  $r^2 = 4 = 3 + 1$  so  $m^2 = 3Q + 1$ , where  $Q = 3q^2 + 2qr + 1$ . In both cases,  $m^2$  is not divisible by 3.

Return to main proof:  $m^2 = 3n^2 \Rightarrow m = 3q$  for some  $q$ . Then  $m^2 = 9q^2 = 3n^2$ , so  $n^2 = 3q^2$ . By the Lemma,  $n$  is divisible by 3. This contradicts our assumption that  $m$  and  $n$  have no common factors. Hence our assumption that  $(m/n)^2 = 3$  is false.