Axioms for the Real Numbers Here are the slightly changed axioms for the reals that we'll use:

R0 The Real Numbers consist of a set, $\mathbb{R}$, and two functions, $+$ and $\times$, each with domain $\mathbb{R} \times \mathbb{R}$ and range $\mathbb{R}$, and the following statements are true:

A1 Given any real numbers a and b, the function $+$ assigns to the ordered pair (a, b) a number we call their *sum,* denoted $a + b$.

A2 $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(a + b = b + a)$. (Commutative Law)

A3 $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})(a + (b + c) = (a + b) + c)$. (Associative Law)

A4 $(\exists z \in \mathbb{R})(\forall a \in \mathbb{R})(a + z = a = z + a)$. (Existence of an additive identity element)

Theorem If $z = z_1$ and $z = z_2$ each satisfy the statement $(\forall a \in \mathbb{R})(a + z = a = z + a)$ then $z_1 = z_2$.
Thus there is exactly one identity element.
Notation: We will write 0 for the unique $z \in \mathbb{R}$ that satisfies the statement $(\forall a \in \mathbb{R})(a + z = a = z + a)$.

A5 $(\forall a \in \mathbb{R})(\exists b \in \mathbb{R})(a + b = 0)$. (Existence of additive inverses)

Theorem If $b = b_1$ and $b = b_2$ each satisfy the equation $a + b = 0$, then $b_1 = b_2$.
Thus, additive inverses are unique.
Notation: We will write $-a$ for the unique b that satisfies the equation $a + b = 0$.

M1 Given any real numbers a and b, the function $\times$ assigns to the ordered pair (a, b) a number we call their *product,* denoted ab, $a \times b$, or $a \cdot b$.

M2 $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(ab = ba)$. (Commutative Law)

M3 $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})a(bc) = (ab)c$. (Associative Law)

M4 $(\exists e \in \mathbb{R})(\forall a \in \mathbb{R})(ae = a = ea)$. (Existence of an multiplicative identity element)

Theorem If $e = e_1$ and $e = e_2$ each satisfy the statement $(\forall a \in \mathbb{R})(ae = a = ea)$ then $e_1 = e_2$.
Thus there is exactly one identity element.
Notation: We will write 1 for the unique $e \in \mathbb{R}$ that satisfies the statement $(\forall a \in \mathbb{R})(ae = a = ea)$.

M5 $(\forall a \in \mathbb{R})($ If $a \neq 0$, then $(\exists b \in \mathbb{R})(ab = 1)$. (Existence of multiplicative inverses of non-zero real numbers)

Theorem If $a \neq 0$, and if $b = b_1$ and $b = b_2$ each satisfy the equation $ab = 1$, then $b_1 = b_2$.
Thus, multiplicative inverses are unique.
Notation: If $a \neq 0$ we will write a^{-1} for the unique b that satisfies the equation $ab = 1$.

N $1 \neq 0$. (Non-triviality axiom)

D $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})(\forall c \in \mathbb{R})\ a(b + c) = ab + ac$. (Distributive Law)

O There exists a set $P \subseteq \mathbb{R}$ with these properties:

(i) Exactly one of the following statements about an arbitrary $x \in \mathbb{R}$ is true:

$$(\alpha)\ \ x \in P; \qquad (\beta)\ \ x = 0; \qquad (\gamma)\ \ -x \in P.$$

(ii) P is closed under addition and multiplication, which means that

$$(\forall a \in P)(\forall b \in P)(a + b \in P \ \text{ and } \ ab \in P) \ \text{ is true.}$$

For the last axiom we need two terms: a set $S \subseteq \mathbb{R}$ is *bounded above* if there exists a real number $U \in \mathbb{R}$ such that for all $s \in S$, it is true that $s \leq U$. In this case, U is called *an upper bound* for (or of) S.

LUB For all subsets S of $\mathbb{R}$, if S is non-empty and S is bounded above then there exists a number $\sigma \in \mathbb{R}$ that is an upper bound for S such that no number less than σ is an upper bound for S.

"In logic" this is:

$$(\forall S \in 2^{\mathbb{R}})\big(S \neq \emptyset \wedge (\exists U \in \mathbb{R})(\forall s \in S)(s \leq U)\big) \Rightarrow (\exists \sigma \in \mathbb{R})\big[(\forall s \in S)(s \leq \sigma) \wedge (\forall x \in \mathbb{R})(x < \sigma \Rightarrow (\exists u \in S)(x < u))\big]$$