Ask! Indicate your approach! Show your work! Good Luck! There are 8 *questions,* 5 *pages,* 110 *points.*

(1) [15] What is the definition of "inner product space?"

An *inner-product space* with complex scalars, $\mathbb{C}$, is a vector space V with complex scalars, and a complex-valued function $\langle v, w\rangle$, called the *inner product,* defined on $V \times V$, that has the following properties:
(a) For all $v \in V$, $\langle v, v\rangle \ge 0$ [3].
(b) If $\langle v, v\rangle = 0$ then $v = 0$ [3].
(c) For all v and w in V, $\langle v, w\rangle = \overline{\langle w, v\rangle}$ [3].
(d) For all v_1, v_2 and w in V, $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$ [3].
(e) For all v, w in V, and all scalars a, $\langle av, w\rangle = a\langle v, w\rangle$ [3].

(2) [15] Define Multiresolution Analysis.

A *Multiresolution Analysis (MRA)* of $L^2(\mathbb{R})$ is a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$ with these properties:

(i) Each V_j, $-\infty < j < +\infty$, is a closed subspace of $L^2(\mathbb{R})$; [3]

(ii) For each j, $V_j \subseteq V_{j+1}$, i.e., the spaces V_j are nested; [3]

(iii) $\bigcap_j V_j = \{0\}$, and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$; [3]

(iv) $f(t) \in V_j \iff f(2t) \in V_{j+1}$; [3]

(v) There is a function $\varphi(t) \in V_0$ such that $\{\varphi(t - n) : n \in \mathbb{Z}\}$ is an o.n. basis for V_0 [3].

(3) [15] Find the projection of $f(t)$ on V_0 (the one using box functions), where $f(t) = \sin \pi t$ when $|t| < 2$ and $f(t) = 0$ elsewhere. Also, find the projection of $f(t)$ onto V_{-1}.

$P_{V_0} f(t) = \sum_k c_k B(t - k)$, where $c_k = \langle f(t), B(t - k)\rangle = \int_k^{k+1} f(t)\, dt$ [5]. Since the support of $B(t - k)$ is the interval $[k, k + 1]$, $\int_k^{k+1} f(t)\, dt = 0$ if $k + 1 \le -2$ or if $k \ge 2$. Thus $c_k \ne 0$ only if $-2 \le k \le 1$. By the way $f(t)$ is defined, we see that $c_{-2} = c_0 = \int_0^1 \sin t\, dt$ and $c_{-1} = c_1 = -c_0$. This means that $P_{V_0} f(t) = c_0(B(t + 2) - B(t + 1) + B(t) - B(t - 1))$. This could have been conveyed with pictures [5].

$P_{V_{-1}} f(t) = 0$ because the averages are taken over intervals of length 2 that have even endpoints [5].

(4) [20] State the Schwarz Inequality and use it to show that $\|v\| := \sqrt{\langle v, v\rangle}$ satisfies the triangle inequality for a norm.

In an inner product space V, for all vectors v, w, $|\langle v, w\rangle| \le ||v||||w||$ [5], and equality holds if and only if one of v and w is a multiple of the other [2].

The triangle inequality is $\|v + w\| \le \|v\| + \|w\|$ [5]. To verify it we have, first,

$$\|v + w\|^2 = \langle v + w, v + w\rangle = \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle = \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2.$$

Thus $\|v + w\|^2 = \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2 = \|v\|^2 + 2\mathbf{Re}\ \langle v, w\rangle + \|w\|^2 \le \|v\|^2 + 2|\langle v, w\rangle| + \|w\|^2$. By the Schwarz inequality, applied to the term with a 2, $\|v + w\|^2 \le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\| + \|w\|)^2$. Taking square roots gives the triangle inequality [10].

(5) [15] Suppose you're given a dyadic interval $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$. How do you tell whether it's the right half or the left half of a dyadic interval $\left[\frac{k'}{2^{j-1}}, \frac{k'+1}{2^{j-1}}\right]$? Which half is $[11/256, 12/256]$?

The dyadic interval $\left[\frac{k'}{2^{j-1}}, \frac{k'+1}{2^{j-1}}\right]$ can be rewritten as $\left[\frac{k'}{2^{j-1}}, \frac{k'+1}{2^{j-1}}\right] = \left[\frac{2k'}{2^j}, \frac{2k'+2}{2^j}\right]$. The left half is $\left[\frac{2k'}{2^j}, \frac{2k'+1}{2^j}\right]$ and the right half is $\left[\frac{2k'+1}{2^j}, \frac{2k'+2}{2^j}\right]$. Therefore $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]$ is the left half if $k = 2k'$, or *if* k *is even,* and it's the right half *if* k *is odd* [10]. Thus $[11/256, 12/256]$ is the right half of $[5/128, 6/128] = [10/256, 12/256]$ [5].

(6) [10] State the Pythagoras Theorem (for inner product spaces) and verify it, noting which properties of inner products that you use.

If $v \perp w$ then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Given: $\langle v, w\rangle = 0 = \langle w, v\rangle$. Then (by expansion steps)
$\|v+w\|^2 = \|v\|^2 + \langle v, w\rangle + \langle w, v\rangle + \|w\|^2 = \|v\|^2 + \|w\|^2$, by what was given.

(7) [10] Define "orthonormal set" and verify that the box functions $B(x-k)$ form an orthonormal set when k runs over the integers.

A set S in an inner product space is *orthonormal* if every element in S has norm 1 and if the inner product of two different elements of S is zero [5].

The box function $B(x-k)$ is zero outside the interval $(k, k+1)$ and is one inside that interval. Thus if $k \neq \ell$, $B(x-k)B(x-\ell) = 0$ for all x. Hence $\langle B(x-k), B(x-\ell)\rangle = \int B(x-k)B(x-\ell)\,dx = 0$ if $k \neq \ell$. Therefore the inner product of two different elements of $\{B(x-k) : k \in \mathbb{Z}\}$ is zero. Finally, $\langle B(x-k), B(x-k)\rangle = \int B(x-k)^2\,dx = \int_k^{k+1} dx = 1$ so every element in $\{B(x-k) : k \in \mathbb{Z}\}$ has norm 1.

(8) [10] Taking $f(t) = 1$ if $0 < t < 1$ and $f(t) = 0$ elsewhere, for which pairs j, k is $\langle f, H_{jk}\rangle \neq 0$? Show your work or explain your answer if you guess it.

$H_{jk}(t) = 0$ outside $\left[\dfrac{k}{2^j}, \dfrac{k+1}{2^j}\right]$ and $f(t) = 0$ outside $(0, 0)$. Thus $\langle f, H_{jk}\rangle = 0$ if $\frac{k+1}{2^j} \leq 0$ (which means $k \leq -1$) or $\frac{k}{2^j} \geq 1$ (which means $k \geq 2^j$). Hence to have $\langle f, H_{jk}\rangle \neq 0$ we need $0 \leq k < 2^j$.

On the other hand if the support of $H_{jk}(t)$ is contained in $[0, 1]$, where $f(t) = 1$, then $\langle f, H_{jk}\rangle = 0$. This happens if and only if $k/2^k \geq 0$ and $k+1 \leq 2^j$. In other words, we need one of $k/2^k < 0$ and $k+1 > 2^j$ to be true. But if $k/2^k < 0$ we know $\langle f, H_{jk}\rangle = 0$. So we need $k+1 > 2^j$ to be true. But we also have to have $0 \leq k < 2^j$. The only way this can be true is to have $k = 0$ and $j < 0$. If that is true, we know that $H_{jk}(t) = 2^{j/2}$ in $(0, 1)$, so that $\langle f, H_{jk}\rangle = 2^{j/2} \neq 0$ if $k = 0$ and $j < 0$.