Ask! Indicate your approach! Show your work! Good Luck! There are 8 questions, 5 pages, 115 points.

(1) [15] Write down the Scaling Equation and the Fourier Scaling Equation, for a MRA. How are these equations found, if we are given a MRA?

(Scaling)
$$\varphi(t) = \sum_n h(n)\sqrt{2}\,\varphi(2t-n), \quad \text{and} \quad \sum_n |h(n)|^2 = 1. \quad [5]$$

(Fourier Scaling)
$$\widehat{\varphi}(\xi) = \sum_n \frac{h(n)}{\sqrt{2}}\, e^{-in\xi/2}\widehat{\varphi}\left(\frac{\xi}{2}\right) =: m_o(\xi/2)\widehat{\varphi}(\xi/2). \quad [5]$$

(Scaling) equation is found from $\varphi \in V_1$, representing φ by its "Fourier series" with respect to the o.n. basis $\{\sqrt{2}\,\varphi(2t-n) : n \in \mathbb{Z}\}$ of V_1, and $h(n) = \left\langle \varphi, \sqrt{2}\,\varphi(2t-n)\right\rangle_{dt}$. (Fourier Scaling) is obtained by applying the Fourier transform to (Scaling). [5]

(2) [10] When we start with assumptions about numbers h_n instead of with a MRA, how do we go about constructing a scaling function? A **brief** answer is sought, that includes a description of what we actually construct.

Our assumptions were $(h1)$–$(h4)$ and they allowed us to define $m_o(\xi) := \sum_n (h_n/\sqrt{2})e^{-in\xi}$ and then show that $\prod_{k=1}^{\infty} m_o(\xi/2^k)$ converges to an L^2 function $\widehat{\Phi}(\xi)$, that gave a function $\Phi(t)$, or maybe $c\Phi(t)$, as a candidate for a scaling function. [10]

(3) [15] When we *start* with a MRA, how do we find the "low-pass filter?" What is its formula? What important equation links certain pairs of values of the "low-pass filter?"

We use the Fourier Scaling Equation (see # 1) [5]
to define $m_o(\xi) := \sum_n (h(n)/\sqrt{2})e^{-in\xi}$. [5]
The equation asked for is $|m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 = 1$ a.e. [5]

(4) [20] Find the Fourier transform of $e^{-2|x|}$.

$$\begin{aligned} I := \int e^{-2|x|}e^{-i\xi x}dx &= \int_{-\infty}^{0} e^{2x}e^{-i\xi x}dx + \int_0^{\infty} e^{-2x}e^{-i\xi x}dx =: I_1 + I_2 \\ &= \int_0^{\infty} e^{-2x}e^{i\xi x}dx + \int_0^{\infty} e^{-2x}e^{-i\xi x}dx, \quad \text{so} \quad I_1 = \overline{I_2}. \end{aligned}$$

$I_2 = \int_0^\infty e^{-x(2+i\xi)}dx = \frac{e^{-x(2+i\xi)}}{-(2+i\xi)}\Big|_0^\infty = \frac{1}{2+i\xi}$, so $I = I_1 + I_2 = \overline{I_2} + I_2 = \overline{\frac{1}{2+i\xi}} + \frac{1}{2+i\xi} = \frac{1}{2-i\xi} + \frac{1}{2+i\xi} = \frac{4}{4+\xi^2}$.

Note: An alternative way is to write $I = \int e^{-2|x|}\cos\xi x\,dx = 2\int_0^\infty e^{-2x}\cos\xi x\,dx$. Then we integrate by parts *twice* to get the cosine back, with a suitable factor:

$$\begin{aligned} I = 2\int_0^\infty e^{-2x}\cos\xi x\,dx &= -2\int_0^\infty \frac{e^{-2x}}{-2}(-\sin\xi x)\xi\,dx + 2\frac{e^{-2x}}{-2}\cos\xi x\Big|_0^\infty \\ &= -\xi\int_0^\infty e^{-2x}\sin\xi x\,dx + 1 \\ &= \xi^2\int_0^\infty \frac{e^{-2x}}{-2}\cos\xi x\,dx + \frac{e^{-2x}}{-2}\sin\xi x\Big|_0^\infty + 1 \\ &= \frac{-\xi^2}{2}\int_0^\infty e^{-2x}\cos\xi x\,dx + 1 = \frac{-\xi^2}{4}I + 1, \quad \text{which gives the same answer.} \end{aligned}$$

(5) [15] Define Multiresolution Analysis.

A *Multiresolution Analysis (MRA)* of $L^2(\mathbb{R})$ is a collection $\{V_j\}$ of subspaces of $L^2(\mathbb{R})$ with these properties:

(i) Each V_j, $-\infty < j < +\infty$, is a closed subspace of $L^2(\mathbb{R})$; [3]

(ii) For each j, $V_j \subseteq V_{j+1}$, i.e., the spaces V_j are nested; [3]

(iii) $\bigcap_j V_j = \{0\}$, and $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$; [1 & 2]

(iv) $f(t) \in V_j \iff f(2t) \in V_{j+1}$; [3]

(v) There is a function $\varphi(t) \in V_0$ such that $\{\varphi(t-n) : n \in \mathbb{Z}\}$ is an o.n. basis for V_0. [3]

(6) [10] State the Pythagoras Theorem (for inner product spaces) and verify it, noting which properties of inner products that you use.

See the solutions for Test 1!

(7) [20] List the 4 equations we started with, about finitely many non-zero numbers, that we used in connection with the Cascade Formula. What did each equation say about m_o (if anything...)?

$(h1)$ $$\sum_{n\in\mathbb{Z}} |h_n|^2 = 1;\ m_o \in L^2(\mathbb{T}); \quad [3\ \&\ 2]$$

$(h2)$ $$\sum_{n\in\mathbb{Z}} h_n = \sqrt{2};\ m_o(0) = 1; \quad [3\ \&\ 2]$$

$(h3)$ $$\sum_{n\in\mathbb{Z}} (-1)^n h_n = 0;\ m_o(\pi) = 0; \quad [3\ \&\ 2]$$

$(h4)$ $$\sum_{n\in\mathbb{Z}} h_n \overline{h_{n-2k}} = 0 \ \text{ if } \ k \neq 0; \quad \text{Used in showing} \quad |m_o(\xi)|^2| + |m_o(\xi+\pi)|^2 = 1. \quad [3\ \&\ 2]$$

(8) [10] Why can't we find three non-zero numbers h_0, h_1, h_3 that satisfy equations $(h1)$–$(h4)$? Here, all other h_n are set equal to zero.

From $(h4)$ here we have $0 = h_0\bar{h}_2 + h_1\bar{h}_3 = h_1\bar{h}_3$ since $h_2 = 0$; but this contradicts $h_1 \neq 0 \neq h_3$.