Assignments are due at the start of class on the due date.
Give credit for help received, including books and hints from me and others; mention discussions.
If a problem is difficult, please include a "narrative" that tells what you went thru to reach your results.

Challenge Problem Due May 12. Prove that $P_N f \to f$ in L^2 as $N \to +\infty$, and that $P_N f \to 0$ in L^2 as $N \to -\infty$.

Assignment 10 Due Apr 25.

1: Suppose that $\widehat{f}(\xi) = F(\xi)\widehat{\varphi}(\xi/2)$ and $\widehat{g}(\xi) = G(\xi)\widehat{\psi}(\xi/2)$, where both of $F(\xi)$ and $G(\xi)$ belong to $L^2(\mathbb{T})$. What is the relationship between $\|f\|^2$ and $\sum_n |F_n|^2$, where $F_n := \frac{1}{2\pi}\int_0^{2\pi} F(\xi)e^{-in\xi}d\xi$? Verify that a similar relationship holds for g and G.

2: Letting P_j denote the projection operator onto the subspace V_j, and Q_j denote the projection operator onto the subspace W_j, verify that $P_j = Q_{j-1} + P_{j-1}$. In what sense is it true that $I = \sum_j Q_j$?

3: Suppose $f \in V_j$, and $r = m/2^{j+n}$, where m is odd and $n > 0$. Find the smallest k such that $f(t+r) \in V_{j+k}$ is true for all $f \in V_j$. Are there *some* $f \in V_j$ that satisfy $f(t+r) \in V_{j+k'}$ for some $k' < k$?

4: Suppose $\zeta(\xi) = -\zeta(\xi + \pi)$, where $\zeta(\xi)$ is 2π-periodic. Suppose also that $\zeta(\xi)\overline{m_o(\xi+\pi)} \in L^2(\mathbb{T})$. Show that $\zeta(\xi) \in L^2(\mathbb{T})$.

Special Problem 3 Due Apr 21

Verify that if $w \in \mathbb{C}^2$ and $z \in \mathbb{C}^2$ and $z \neq 0$ then $w \perp z$, as vectors in $\mathbb{C}^2$, if and only if there exists a complex number ζ such that

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \zeta \begin{pmatrix} -\overline{z_2} \\ \overline{z_1} \end{pmatrix}.$$

Assignment 9 Due Apr 16.

1: Suppose we want h_n satisfying $(h4)$, with $h_n = 0$ unless $0 \le n \le N-1$, with $h_0 \neq 0 \neq h_{N-1}$. Show that if $(h4)$ is true for our h_n then N has to be even. Show that if $(h4)$ is true for some $k > 0$ then $(h4)$ is true for $-k$ as well. Find the number of equations that $(h4)$ produces. Hint: For $k>0$, verify that $(h4)$ becomes $\sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}} = 0$.

2: Use L^2 Fourier Facts to verify that if $f \in L^2$ and $g \in L^2$ then $\int f(t+x)\overline{g(t)}\,dt = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$.

3: Use Minkowski's Integral Inequality (Lebesgue Facts) to verify that if $f \in L^1$ and $g \in L^2$ then $f * g \in L^2$. Hint: $f*g = g*f$.

4: Write out the relevant $(h4)$ equations, as in # 1, when $N = 6$. Set $\lambda := -h_5/\overline{h_0}$. How far does this get us in finding solutions for the $(h4)$ equations? What if we just want *real-valued* h_n's?

Assignment 8 Due Apr 9. These problems are easy to solve, using "Lebesgue Facts."

1: Show that the set of all $f \in L^2(\mathbb{R})$ such that $f(t) = 0$ a.e. in (M_1, M_2) is a closed subspace of $L^2(\mathbb{R})$. We used this when we proved that our $\Phi(t)$ has compact support contained in $[M_1, M_2]$.

2: Show that if $f(t) \in L^1(\mathbb{R})$ then $\widehat{f}(\xi)$ is a continuous function of ξ. We used this to evaluate $\widehat{\Phi}(0)$ *as an integral.*

3: Show that if $f(t) \in L^1(\mathbb{R})$ and $g(t) \in L^1(\mathbb{R})$ then their convolution, $f * g(t) := \int f(t-s)g(s)\,ds$, is also in $L^1(\mathbb{R})$ and $\|f*g\|_1 \le \|f\|_1\|g\|_1$.

4: Show that if $f(t) \in L^1(\mathbb{R})$ and $g(t) \in L^1(\mathbb{R})$ then $\widehat{(f*g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$ for all ξ.

Assignment 7 Due Apr 2

1: We can "define" a wavelet $\psi(t)$ corresponding to a scaling function $\varphi(t)$, using the Fourier transform, by means of the equation $\widehat{\psi}(\xi) := e^{i\xi/2}\overline{m_o(\xi/2+\pi)}\widehat{\varphi}(\xi/2)$. Working backwards from this equation, find the coefficients d_k that make the equation $\psi(t) = \sum_k d_k\sqrt{2}\varphi(2t-k)$ true. The numbers d_k are expressible in terms of the h_n's.

2: Carry out the steps in # 1 when φ is the Box function, and find both of $\widehat{\psi}$ and ψ.

3: Find the *convolution,* $g * h(x) := \int g(x-y)h(y)\,dy$, when $g(x) = e^{-|x|} = h(x)$, and find the Fourier transform of $g * h(x)$. Looking at "Fourier Facts" and Assgt 6 may help!

Special Problem 2 Due Mar 31

(a) Suppose that $h_n = 0$ unless $0 \le n \le 2K-1$, where K is a positive integer. Also suppose that $\{h_n\}$ satisfies $(h1)$–$(h4)$. Verify that for all $k \in \mathbb{Z}$,

$$(*) \qquad \sum_{n\in\mathbb{Z}} (-1)^n h_n h_{-n-2k-1} = 0.$$

(b) Verify that $(*)$ holds for *every* sequence $\{h_n\} \in \ell^2$, not just those of part (a).
Hints: What happens in (a) if $k \ge 0$? $k=-1$? $k=-2$? Do (b) first?

Assignment 6 Due Mar 26

1: We define numbers h_n as follows.
Let $16\sqrt{2}\,h_0 = -3+\sqrt{15}$, $16\sqrt{2}\,h_1 = 1-\sqrt{15}$,
$8\sqrt{2}\,h_2 = 3-\sqrt{15}$, $8\sqrt{2}\,h_3 = 3+\sqrt{15}$,
$16\sqrt{2}\,h_4 = 13+\sqrt{15}$, $16\sqrt{2}\,h_5 = 9-\sqrt{15}$.
Let $h_n = 0$ for all other n. Show that these 6 coefficients satisfy equations $(h1)$–$(h4)$.

2: Using the numbers h_n in # 1, find out whether or not $m_o(\xi) = 0$ is true for some ξ with $|\xi| \le \pi/2$. The use of some computer software may be valuable here. If you do use software, include "source code" as well as output!

3: Find the Fourier transform of the function $f(t)$ equal to e^{-t} if $x > 0$ and equal to zero otherwise. *Without* computing the integral again, find the Fourier transform of $f(-t)$. Verify that $\widehat{f}(\xi) \in L^2(\mathbb{R})$, but that $\widehat{f}(\xi) \notin L^1(\mathbb{R})$.

4: Find the Fourier transform of the function $f(t) = e^{-|t|}$. Verify that $\widehat{f}(\xi) \in L^1(\mathbb{R})$. You might find that you can use your work in # 3 to avoid computing an integral.

Assignment 5 Due Mar 12

1: Why do equations $(h1)-(h4)$ have no solutions with just one non-zero $h(n)$? What about exactly three non-zero $h(n)$'s?

2: Find all solutions of $(h1)-(h4)$ with exactly two non-zero $h(n)$'s.

3: Find the Fourier transform of the *even* function $f(x)$ that is equal to $1-x$ when $0 \le x \le 3/2$, equal to $x-2$ when $3/2 \le x \le 2$ and equal to zero when $x \ge 2$. Simplify your formula and put it over a common denominator. Sketch $\widehat{f}(\xi)$. Suggestions: The Fourier transform of an even function uses $2\cos\xi x$ on $(0,\infty)$ instead of $\exp(-i\xi x)$ on $(-\infty,\infty)$. First find $\hat{f}(0)=\int f$, then integrate by parts when $\xi\neq 0$.

4: Try to construct a function $\varphi(t)$ that is piecewise linear and continuous (the function $f(x)$ in # 3 is such a function) and such that $\varphi(t)$ is a scaling function for an MRA. First, state what is needed. Say what goes wrong if we look for a $\varphi(t)$ of this type that has compact support (i.e., is zero outside some bounded interval). Don't actually *construct* such a $\varphi(t)$; just give directions for constructing one.

Assignment 4 Due Mar 5

1: # 1.4, pp 15–16 in the text (also, read 1.1, 1.2 and 1.9).

2: # 2.5, pp 44 in the text; the first "Show that."

3: # 2.5, pp 44 in the text; the second "Show that."
(For 2: and 3:, note that the author suppresses the $\sqrt{2}$ that I used in class).

4: Suppose that $\mathcal{O}$ is an orthonormal set in a Hilbert space H. Suppose that, for all $v \in H$, Parseval's Relation ((19) in the Hilbert space notes) holds. Show that $\mathcal{O}$ is an orthonormal basis for H.

Assignment 3 Due Feb 26

1: Find the Haar coefficients c_{jk} for the function $v(x)$ that is equal to one in $(1, 2)$, equal to negative one in $(2, 3)$ and equal to zero elsewhere. Express the function as a Haar series and verify that the sum of the squares of the coefficients is equal to $\|v\|^2$.

2: Find the Haar coefficients c_{jk} for the function $w(x)$ that is equal to one in $(-1, 0)$, equal to negative one in $(0, 1)$ and equal to zero elsewhere. Express the function as a Haar series and verify that the sum of the squares of the coefficients is equal to $\|w\|^2$. How does this compare to the series for $v(x)$?

3: Find the Fourier transform of the Box function $B(x)$ ($= 1$ in $(0, 1)$ and $= 0$ elsewhere) and verify that the equation $\widehat{B}(\xi) = \left\{ \sum_k \frac{h(k)}{\sqrt{2}} e^{-ik\xi/2} \right\} \widehat{B}(\xi/2)$ is true, where $h(0) = 1/\sqrt{2} = h(1)$ and $h(k) = 0$ for all other k.

Assignment 2 Due Feb 19

1: The equation $V_1 = V_0 \oplus W_0$ says that we can write every element of V_1 as the sum of an element in V_0 and an element of W_0. Verify that this can be done in only one way. Decompose V_0 in the same way. How are W_0 and W_{-1} related to each other (it's similar to the way V_0 and V_{-1} are related). This will give us an equation $V_1 = (V_{-1} \oplus W_{-1}) \oplus W_0$. Explain why it's OK to write this as $V_1 = V_{-1} \oplus W_{-1} \oplus W_0$, and say what the equation means. Generalize (without details) to equations of the form $V_j = W_{j-1} \oplus W_{j-2} \oplus \cdots \oplus W_{j-m} \oplus V_{j-m}$, and say what the equation means.

2: Describe the projection operator $P_{V_0} : V_1 \to V_1$. Here, $H = V_1$ and $X = V_0$; you're being asked to say, given $v_1 \in H = V_1$, how to calculate $P_X v_1 = P_{V_0} v_1$. Do the same for $P_{X^\perp}$ in this context.

3: In Problem 4 of Assignment 1 you found the coefficients c_{jk} of the functions $u(x)$ equal to one when $-1 < x < 1$ and equal to zero elsewhere. Calculate the sum of the squares of all your coefficients. This will be the series $\sum_{j \in \mathbb{Z},\ k \in \mathbb{Z}} c_{jk}^2$. The vast majority of your coefficients are zero, but there are still infinitely many that are not zero.

4: Consider the function $u(x)$ in problem 4 of Assignment 1. To which of the spaces V_j does $u(x)$ belong? To which of the spaces $V_j^\perp$ does $u(x)$ belong? To which of the spaces W_j does $u(x)$ belong? To which of the spaces $W_j^\perp$ does $u(x)$ belong? Find a finite set F_ϵ of the coefficients c_{jk} that you calculated so that $\sum_{c_{jk} \in F_\epsilon} c_{jk} H_{jk}(x)$ is closer (in L^2 norm) to $u(x)$ than ϵ, where ϵ takes the values $1/2$, $1/8$, $1/64$. Sketch your approximations as well.

Assignment 1 Due Feb 10

1: Suppose M is the 4×4 matrix with columns $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 4 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 2 \\ 1 \end{pmatrix}$, which we name $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, respectively. Given a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, where the x_j are complex numbers, say how to express $\mathbf{x}$ as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. That is, say how to find c_1, c_2, c_3, c_4 such that $\mathbf{x} = \sum_{j=1}^4 c_j \mathbf{v}_j$. In particular, find the c_j's such that $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \sum_{j=1}^4 c_j \mathbf{v}_j$.

2: Suppose that M is an $n \times n$ matrix of complex numbers with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, respectively. Suppose also that the "conjugate transpose," $\overline{M^T}$, of M is also the inverse of M. "Do" Problem 1 with this M, and compare. (Your explicit vector will now be the *column* version of $(1, 2, \ldots, n)$).

3: For $x \in \mathbb{R}$, define $H(x) := 1$ if $0 < x < 1/2$, $H(x) := -1$ if $1/2 < x < 1$, and let $H(x) := 0$ for all other x.
(i) Find j, k so that $H_{jk} = H$.

(ii) Shetch the graphs of $H_{2,1}$, $H_{-2,1}$, and $H_{3,-1}$.
(iii) Verify that the functions $H_{j,k}(x) = 2^{j/2}H(2^j x - k)$ satisfy

$$\langle H_{jk},\, H_{j'k'}\rangle = \begin{cases} 1 & \text{if } j' = j \text{ and } k' = k \\ 0 & \text{if } j' \neq j \text{ or } k' \neq k. \end{cases}$$

We say the H_{jk} *form an orthonormal system of functions in* $L^2(\mathbb{R})$ (these are the Haar functions; H is the Haar wavelet).

4: In Problem 3, j refers to "scale," and k refers to "position." Find the Haar coefficients of the function $u(x)$ that is equal to 1 for $-1 < x < 1$ and equal to zero elsewhere. That is, find

$$c_{jk} := \langle u,\, H_{jk}\rangle = \int_{-\infty}^{\infty} u(x)\overline{H_{jk}(x)}\, dx,$$

where the complex conjugation can be ignored since each H_{jk} is real-valued. I am particularly interested in how you arrange this two-parameter family of coefficients. One thought you might bear in mind is to compare your arrangement to sheet music.

Special Problem 1 Due Feb 3

Carry out the verification of items (7) (8) and (10) in the Hilbert space notes (i.e., use expansion steps and the "rules" for an inner product space to check the formulas).