Assignments are due at the start of class on the due date.
Give credit for help received, including books and hints from me and others; mention discussions.
If a problem is difficult, please include a "narrative" that tells what you went thru to reach your results.

Special Problems are Term Papers! They must be written neatly, not on paper torn from a spiral notebook, and they *must* have one-inch margins on all four sides! The top margin on the first page, though, is for your name and the Assignment title. Writing on both sides is fine. If your handwriting fills the space between two lines on lined paper, double-space! Points are very likely to be deducted for these things and others, such as prolixity and failure to cite references and help from others.

Assignment 14 Due May 6:

Project (worth 3 Special Problems)

Assignment 13 Due May 4:

#1: Turn in an Abstract of your Project and a sample of output. If output is not yet ready, a sketch is OK.

Assignment 12 Due Apr. 27:

#1: In the *general* context, show that $W_j \perp V_j$ for all $j \in \mathbb{Z}$.
#2: Letting P_j denote the projection operator onto the subspace V_j, and Q_j denote the projection operator onto the subspace W_j, verify that $P_j = Q_{j-1} + P_{j-1}$.
#3: Working with $\psi(t) := B(2t) - B(2t-1)$ and $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$ find, for each j, which of the $\psi_{jk}(t)$ (if any!) have support in $[0, 1]$.
#4: (continues #3): show that the collection of all of the $\psi_{jk}(t)$ that have support in $[0, 1]$, together with the Box function, is an orthonormal basis for $L^2(0, 1)$.

Assignment 11 Due Apr. 20:

#1: In the *general* context, show that $W_j \perp W_\ell$ if $j \neq \ell$.
#2: A counting problem. Suppose that $f \in V_J$, and that $f = \sum_{k=K_1+1}^{K_1+2^m} \langle f, \varphi_{Jk}\rangle \varphi_{Jk}$. When we write $f = g_1 + f_1$, where $g_1 \in W_{J-1}$ and $f_1 \in V_{J-1}$, how many coefficients *may* be needed to express g_1 in terms of the $\psi_{J-1,k}$? How many coefficients *may* be needed to express f_1 in terms of the $\varphi_{J-1,k}$?
#3 (continues #2): If the process started in #2 is continued, so that after N steps we have $f = g_1 + \cdots + g_N + f_N$, how many coefficients *may* be needed? If N is large enough then because we started with a *finite* swath of coefficients, f_N is eventually 0. What is the *first* N such that $f_N = 0$? Why?
#4: Given: $z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a non-zero vector in $\mathbb{C}^2$, $w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is also in $\mathbb{C}^2$ and $w \perp z$ in $\mathbb{C}^2$. We let $\langle w, z\rangle$ denote their inner product in $\mathbb{C}^2$, namely $\langle w, z\rangle = w_1\overline{z_1} + w_2\overline{z_2}$. Show that $w \perp z$ in $\mathbb{C}^2$ if and only if $w = \zeta \begin{pmatrix} -\overline{z_2} \\ \overline{z_1} \end{pmatrix}$, where ζ is a (complex) constant of proportionality.
Suggestion: Assume $\langle w, z\rangle = 0$ and then solve for w_1 and w_2.

Assignment 10 Due Apr. 13: The only problem on this assignment: Prove this statement, that we have used, and will be using again soon: *If* H *is a Hilbert space and* $S \subseteq H$ *then* span(S) *is dense in* H *if and only if it is true that:* $y \perp S \iff y = 0$. This is in part a **review** problem: you'll probably have to review the Hilbert Space notes and definitions such as "dense..." Questions are welcome!

Assignment 9 Due Apr. 6:

#1: Define $$R(t) := \begin{cases} 0 & \text{if } t \le 0; \\ -t & \text{if } 0 < t \le \pi; \\ -3\pi + 2t & \text{if } \pi < t \le 2\pi; \\ 5\pi - 2t & \text{if } 2\pi < t \le 3\pi; \\ -4\pi + t & \text{if } 3\pi < t \le 4\pi; \\ 0 & \text{if } t > 4\pi. \end{cases}$$
Sketch $R(t)$. Periodize $R(t)$. Sketch the resulting 2π-periodic function $P(t)$. Find the Fourier coefficients of $P(t)$.

#2: (continues Assgt 7, #2) Define (for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$) $\psi_{jk}(t) := 2^{j/2}\psi(2^j t - k)$. Show that for each j and k, $\psi_{jk} \in V_{j+1}$ but $\psi_{jk} \notin V_\ell$ if $\ell < j+1$. Also, find the smallest interval $[a_{jk}, b_{jk}]$ *outside of which* $\psi_{jk}(t) = 0$.
#3: (continues #2) Show that $\{\psi_{jk} : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ is an orthonormal set

Assignment 8 Due Mar. 30:

#1: Show that the set of all $f \in L^2(\mathbb{R})$ such that $f(t) = 0$ a.e. in (M_1, M_2) is a closed subspace of $L^2(\mathbb{R})$. We used this when we proved that our $\Phi(t)$ has compact support contained in $[M_1, M_2]$.

#2: Show that if $f(t) \in L^1(\mathbb{R})$ then $\widehat{f}(\xi)$ is a continuous function of ξ. We used this to evaluate $\widehat{\Phi}(0)$ *as an integral.*

#3: Suppose we want h_n satisfying $(h4)$, with $h_n = 0$ unless $0 \le n \le N-1$, with $h_0 \ne 0 \ne h_{N-1}$. Show that if $(h4)$ is true for our h_n then N has to be even. Show that if $(h4)$ is true for some $k > 0$ then $(h4)$ is true for $-k$ as well. Find the number of equations that $(h4)$ produces. Hint: For $k>0$, verify that $(h4)$ becomes $\sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}} = 0$.

#4: Use L^2 Fourier Facts to verify that if $f \in L^2$ and $g \in L^2$ then $\int f(t+x)\overline{g(t)}\,dt = \frac{1}{2\pi}\int e^{i\xi x}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$.

Special Problem 4: Due Mar. 25

Show that there are no solutions of $(h1)$ – $(h4)$ that have just three non-zero h_n's. Suggestion: begin by assuming the non-zero ones are consecutive h_n's.

Assignment 7 Due Mar. 23:

#1: Supposing that h_0, h_1, h_2, h_3 are real and that all other h_n are zero, find a formula for h_1 in terms of h_0 that allows you to find all solutions, for such h_n's, of $(h1)$ – $(h4)$. Your solutions should include the ones with just two non-zero h_n's among h_0, h_1, h_2 and h_3.

#2: Context: the MRA determined by the Box function. *Define* $\psi(t) := B(2t) - B(2t-1)$. To which V_j does ψ belong? Show that $\mathcal{W} := \{\psi(t-n) : n \in \mathbb{Z}\}$ is an orthonormal set.

#3: (continues #2) Find the formula for $\widehat{\psi}(\xi)$. Simplify! Write out the series in Theorem $(II.2)$ in the MRA notes with your formula. For which ξ is the formula true? Why?

#4: (continues #2) Show that $W_0 := \overline{\text{span}(\mathcal{W})} = V_0^\perp \cap V_1$ and that $\mathcal{W}$ is an orthonormal basis for W_0.

Assignment 6 Due Mar. 9: In previous problems you have completed the proof that the space V_0 given in terms of the Box function (as the scaling function) gives an MRA. This assignment asks you to use some of the material you have seen to develop another example of a scaling function, one that does not have compact support. It is also true that this φ is not in L^1.

#1: *Define* $\widehat{\varphi}(\xi) := B\left(\frac{\xi}{2\pi} - \frac{1}{2}\right)$. Use the Fourier Inversion formula, (12) in *Fourier Transform Facts:* L^2 to find the formula for $\varphi(t)$. How do we know that $\varphi \in L^2(\mathbb{R})$? *Assuming* that the Fourier Scaling equation is true, find $m_o(\xi)$.

#2: Use $(II.2)$ in the MRA to show that $(II.1)$(v) is true for φ.

#3: *Assuming* that the Fourier Scaling equation is true, find $m_o(\xi)$.

#4: Using $m_o(\xi)$, found in #3, find the $h(n)$. Which of the equations $(h1)$ – $(h4)$ are true?

Special Problem 3: Due Mar. 4

Show that, with our example of V_0 given in terms of the Box function, $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$.

Assignment 5 Due Mar. 2:
#1: Show that, with our example of V_0 given in terms of the Box function, $\bigcap_j V_j = \{0\}$.

#2: Given that $f \in L^2(0,\, 2\pi)$ and $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, find the Fourier series for $f(x-d)$, d a real constant.

#3: Find the Fourier transform of $2^{j/2}B(2^j t - n)$, $j \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

#4: *Derive* the formula for the $m_o(\xi)$ associated with the Box function and verify *directly* that $|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1$ for all ξ (page 2 of MRA notes, just after $(h2)$) .

Assignment 4 Due Feb. 23: As #1 and #2, the Exercises at (10) and (11) in *Fourier series,* v1 2/11/05; as #3 the (new) Exercise at (16.1) in *Intro. to inner product and Hilbert spaces.*

Special Problem 2: Due Feb 18

The Problem at (9) in *Fourier series,* v1.

Assignment 3 Due Feb. 16: As #1, #2 and #3, the Exercises at (5), (6) and (8) in *Fourier series,* v1 2/7/05; as #4 the (new) Exercise at (17.1) in *Intro. to inner product and Hilbert spaces.*

Assignment 2 Due Feb. 9: (For #1 and #2) In the document with link *Introduction* 1/16/05: (#1) the Exercise with label (1); (#2) the Exercise with label (4); (#3) Suppose $0 < b < 2\pi$. Find the Fourier coefficients of the function that is equal to 1 in $(0,\, b)$, equal to 0 in $(b,\, 2\pi)$ and periodic with period 2π. (#4) What functions would we use as an orthonormal basis for $H = L^2(0,\, P)$, where $P > 0$ and the functions in H are periodic of period P?

Special Problem 1: Due Feb 4

Find the formula analagous to the Polarization Formula when the scalars are real. Check that your formula is correct.

Assignment 1 Due Feb. 2: In *Intro. to inner product and Hilbert spaces,* Exercise (1.1), p1; (12.1), p5 (counts as two problems); (13.1), p5 (counts as two problems).

Assignment 0 No due date

Work the Exercises in the Introduction.