Ask! Indicate your approach! Show your work! Good Luck! There are 5 pages, and 105 points.

(1) [15] Define *differentiable in the complex sense.* With $z = x + iy$, determine for which z each of the following functions is differentiable in the complex sense, and explain how you did so:

(a) $f(z) := 3x^2 - 8xy - 3y^2 + 2i(2x^2 + 3xy - 2y^2)$, (b) $g(z) := x^2y^3 - ix^3y^2$.

$f(z)$ is differentiable in the complex sense at z_o if $\lim\limits_{z \to z_o} \dfrac{f(z) - f(z_o)}{z - z_o}$ exists.

To answer (a) and (b) we use the Cauchy-Riemann equations. Each function is a polynomial so the partials will all be contunuous everywhere.

In (a) $u_x = 6x - 8y$ and $v_y = 6x - 8y = u_x$; $u_y = -8x - 6y$ and $v_x = 8x + 6y = -u_y$ so f is differentiable everywhere.

In (b) $u_x = 2xy^3$ and $v_y = -2x^3y$ so $u_x = v_y$ if (and only if) $xy = 0$. Then $u_y = 3x^2y^2$ and $v_x = -3x^2y^2$ so $u_y = -v_x$ everywhere. Hence g is differentiable on the axes.

(2) [10] Define *analytic function.* For each function in the previous problem, give the set of all points at which the function is analytic.

A function is analytic at a point if it is differentiable in a neighborhood of the point. In # 1, f is entire and g is analytic nowhere.

(3) [15] Define *limit point* and *boundary point.* Sketch the following sets and identify their limit points and boundary points.

(a) $\left\{ \dfrac{n^2 + in}{n^2 + 1} : n \in \mathbb{Z} \right\}$, (b) $\{(x, y) : x > 0,\ y > 0$ and $xy = 1/n$ with $n \in \mathbb{Z}^+ \}$.

z_o is a limit point of S if every neighborhood of z_o contains $z \neq z_o$ with $z \in S$.

z_o is a boundary point of S if every neighborhood of z_o contains a point of S and a point of S^c. Every point of the sets in (a) and (b) is a boundary point. In addition, (a) has boundary point 0, its only limit point. The set in (b) has the non-negative axes as boundary points as well. Moreover, the set of limit points of that set coincides with is set of boundary points.

(4) [15] Find the square roots of i and the fourth roots of i. Your answers must be in radical form, not trigonometric (though using trigonometry is OK at first). No need to simplify the radicals.

Recall: $$\sqrt{z} = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\,\mathrm{sgn}(y)\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right) \text{ and } z = i:$$

$$\sqrt{i} = \pm\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) \text{ and } \sqrt[4]{i} = \pm\sqrt{\pm\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)}. \text{ Here the two } \pm \text{ are independent.}$$

Thus two of the roots are

$$\pm\sqrt{\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)} = \pm\left(\sqrt{\frac{1+\sqrt{\frac{1}{2}}}{2}} + i\sqrt{\frac{1-\sqrt{\frac{1}{2}}}{2}}\right)$$

and the other two are obtained by multiplying by i, one of the square roots of -1:

$$\pm\sqrt{\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right)} = \pm\left(i\sqrt{\frac{1+\sqrt{\frac{1}{2}}}{2}} - \sqrt{\frac{1-\sqrt{\frac{1}{2}}}{2}}\right).$$

(5) [10] State the *Cauchy-Riemann equations* and how the relate to differentiability in the complex sense.

$u_x = v_y$ and $u_y = -v_x$. If the Cauchy-Riemann equations hold at a point z_o and the partials of u and v are continuous in a neighborhood of z_o then $f = u + iv$ is differentiable in the complex sense at z_o.

(6) [10] Define *open set* and state a Theorem giving a different, equivalent, way to tell that a set is open.

A set is open if every point of the set has a neighborhood contained in the set. A set is open if and only if its complement is closed (if and only if it contains none of boundary points).

(7) [10] Define *closed set* and state a Theorem giving a different, equivalent, way to tell that a set is closed.

A set is closed if every boundary point of the set is in the set. A set is closed if and only if its complement is open (if and only if it contains all of its limit points).

(8) [10] Define *harmonic function* and *conjugate harmonic functions.* Verify that $e^x \cos y$ and $e^x \sin y$ are harmonic conjugates.

A function u is harmonic if $u_{xx} + u_{yy} = 0$. A pair of functions u and v are harmonic conjugates if both are harmonic and if u and v satistfy the Cauchy-Riemann equations.

Let $u = e^x \cos y$ and $v = e^x \sin y$. Then $u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$. Same idea for v. $u_x = e^x \cos y = v_y$. $u_y = -e^x \sin y = -v_x$.

(9) [10] State the *Ratio Test.* Show that $\exp(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all z. State a Theorem justifying the statement that $\exp(z)$ is analytic in $\mathbb{C}$.

If $\sum a_n$ has terms eventually non-zero and $L = \lim \frac{|a_{n+1}|}{|a_n|}$ exists and $L < 1$ then $\sum a_n$ converges (absolutely) but if $L > 1$ $\sum a_n$ diverges. If $L = 1$ the test fails. Alternate: substitute $L = \limsup \frac{|a_{n+1}|}{|a_n|}$.

For the exp series $\frac{|a_{n+1}|}{|a_n|} = \frac{|z|}{n_1} \to 0 < 1$ so the series converges absolutely everywhere.

Theorem: If a power series converges to $f(z)$ for $|z| < R$ then f is differentiable in the complex sense for $|z| < R$.

Here R can be as large as we wish so $\exp(z)$ is analytic in $\mathbb{C}$.

Note: The answers given here are a little bit more complete than would have been accepted on the Test. See me for more complete answers!