Ask! Indicate your approach! Show your work! Good Luck! There are 5 pages, and 105 points.

(1) [15] *State the Cauchy-Goursat Theorem. Suppose that D is a domain and that C_1 and C_2 are simple closed curves in D, with C_2 lying inside C_1. Suppose that f is analytic in D except at some points lying inside C_2. Explain why* $\displaystyle\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$.

If C is a *simple* closed curve and f is analytic on and inside C, then $\int_C f(z)\,dz = 0$.

We can use a simple polygonal path inside the region between C_1 and C_2 to connect a point p on C_2 (for example a point on C_2 that is closest to C_1) to a point p' on C_1 that it is closest to. We then do the same at a point q on C_2 that is diametrically opposite p and connect q similarly to a point q' on C_1. These two paths can be chosen so that they divide the region between C_1 and C_2 into two simply connected regions. *The preceding could be done on the test by drawing a picture.* Then we can apply Cauchy-Goursat to the bounday curves of two regions, since the curves are simple and f is analytic on and inside the curves. Since the integrals on the polygonal paths we introduced cancel, the result follows.

(2) [10] *Let C be the curve bounding a square with edges of length six, center zero, traced counterclockwise about its center. Find* $\displaystyle\int_C \frac{2\zeta-1}{\zeta(\zeta-1)}\,d\zeta$.

The function $f(z) := \frac{2z-1}{z(z-1)}$ is not analytic inside the square. But if we rewrite f using partial fractions we have $f(z) = \frac{-1}{z} + \frac{1}{z-1}$ so that $\int_C f(z)\,dz = -\int_C \frac{dz}{z} + \int_C \frac{dz}{z-1} = -\int_C \frac{dz}{z} - \int_C \frac{dz}{1-z} = -2\cdot 2\pi i = -4\pi i$ by the Cauchy Integral Formula applied to the function $f(z) = 1$.

(3) [15] *Find* $\displaystyle\int_C \frac{\bar\zeta}{\zeta - z}\,d\zeta$, *where C is the unit circle, and z is a complex number that does not lie on the unit circle.* Hints: Two cases; consider using a geometric series and uniform convergence in each case.

If z is *inside* the unit circle, $\frac{\bar\zeta}{\zeta-z} = \frac{\bar\zeta}{\zeta}\frac{1}{1-(z/\zeta)} = \frac{\bar\zeta}{\zeta}\sum_{n=0}^{\infty}(\frac{z}{\zeta})^n$ and the series converges uniformly.
Thus $\int_C \frac{\bar\zeta}{\zeta-z}\,d\zeta = \sum_{n=0}^{\infty}\int_0^{2\pi}\frac{e^{-i\theta}}{e^{i\theta}}z^n e^{-in\theta} ie^{i\theta}\,d\theta = \sum_{n=0}^{\infty} z^n \int_0^{2\pi} e^{-i(2+n-1)\theta} i\,d\theta = 0$ since every term in the series is zero because the exponent under the integral sign is the non-zero integer $n+1$ in each term. If z is *outside* the unit circle then $\zeta - z \neq 0$ when $|\zeta| \le 1$ so $\frac{\bar\zeta}{\zeta-z} = \frac{1/\zeta}{\zeta-z} = \frac{1/(\zeta-z)}{\zeta}$ on C. Our integral then has the form $\int_C \frac{f(\zeta)}{\zeta-0}\,d\zeta$, where f is analytic inside and on C. Hence by the Cauchy Integral Formula $\int_C \frac{\bar\zeta}{\zeta-z}\,d\zeta = 2\pi i/(-z)$.

Alternate solution: We can write $\bar\zeta = 1/\zeta$ since we are on the unit circle. Then $\frac{1}{\zeta(\zeta-z)} = \frac{1}{-z}(\frac{1}{\zeta} - \frac{1}{\zeta-z})$. We can apply the Cauchy Integral Formula or Cauchy's Theorem in both cases, except the case $z=0$. If $z=0$ we can calculate directly: $\int_C \frac{1}{\zeta^2}\,d\zeta = \int_0^{2\pi} e^{-2i\theta} ie^{i\theta} d\theta = \int_0^{2\pi} e^{-i\theta} i\,d\theta = 0$. Otherwise, the Cauchy Integral Formula gives $\frac{1}{-z}(2\pi i - 2\pi i) = 0$ if z is inside the circle. If z is outside the circle, the Cauchy Integral Formula applied to the first term $(f(\zeta) := \frac{1}{-z})$ and Cauchy's Theorem applied to $f(\zeta) := \frac{1}{-z}(\frac{-1}{\zeta-z})$ gives 0, so $\int_C \frac{\bar\zeta}{\zeta-z}\,d\zeta = \frac{2\pi i}{-z}$.

(4) [15] *Let C be the curve given parametrically by*

$$\begin{aligned} x &= 4\cos t - 3\cos 2t, \\ y &= 4\sin t - 3\sin 2t, \end{aligned} \qquad -\pi \le t \le \pi:$$

Find $\displaystyle\int_C \frac{d\zeta}{\zeta - z}$, *where z is a complex number that does not lie on C. The value will depend on z. Use Theorems to "calculate" this integral, and identify them clearly. What are the answers for* $\displaystyle\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$ *when f is analytic in an open disc that contains C?*

No points were deducted if you did not answer the question at the end. The integral can be expressed as the integral over the outer loop minus the integral over the inner loop, both integrals taken in the positive direction. This gives two integrals over simple closed curves instead of the non-simple curve shown. Then by the Cauchy Integral Formula we have $2\pi i(f(z) - f(z)) = 0$ if z is inside the inner loop. If z is inside the outer loop but outside the inner loop, we use the Cauchy Integral Formula on the outer integral and Cauchy's Theorem on the inner-loop integral to get $2\pi i f(z) - 0 = 2\pi i f(z)$. If z is outside both loops the integrand is analytic inside and on the curve C so Cauchy's Theorem gives 0 again as the value of the integral.

(5) [10] *State the Cauchy Integral Formula.* **Briefly,** *describe how we used the "triangle" version of the Caucy-Goursat Theorem to derive it.*

If f is analytic on and inside a simple closed curve C, then for all z *inside* C $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$. To prove this we drew a small triangle inside C that included z in its interior. The function $g(\zeta)$ defined to be $\frac{f(\zeta)-f(z)}{\zeta - z}$ when $\zeta \neq z$ and defined to be $f'(z)$ when $\zeta = z$ is analytic in the domain inside C except possibly at z, and is continuous at z because f is (complex) differentiable at z. Then the integral of $g(\zeta)$ about the triangle is zero, and algebraic manipulations yield the formula. Finally we use deformation of contours to see that the integral about the curve C is the same as the integral about the triangle. *Note:* We only *proved* the full Cauchy-Goursat Theorem for *convex* domains! However, it is true for domains surrounded by a simple closed curve.

(6) [10] *Calculate directly* $\displaystyle\int_C \frac{dz}{z}$ *when* C *is a simple closed curve, of your choice, that encloses the origin.*

We choose C to be the unit circle. On C, $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then $dz = z'(\theta)\,d\theta = ie^{i\theta}\,d\theta$ so
$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = 2\pi i$.

(7) [10] *Find* $\displaystyle\int_C \frac{dz}{z}$ *when* C *is an arbitrary simple closed curve that encloses the origin. Explain your method!*

We may use the Cauchy Integral Formula with $f(z) = 1$ for all z and put $z_o = 0$.
Then $1 = f(z_o) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_o}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_C \frac{dz}{z}$.

(8) [10] *What is the formula for the radius of convergence of a power series?*

Find the radius of convergence of each series: (a) $\displaystyle\sum_{n=0}^{\infty} \frac{z^{n^2}}{n^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n^n}$.

The *radius of convergence* R of a power series $\sum c_n z^n$ is given by $1/R = \limsup_{n\to\infty} |c_n|^{1/n}$.

(a) If n is not a square, $|c_n|^{1/n} = 0$. When we consider n^2, $c_{n^2} = 1/n^n$.
Then $|c_{n^2}|^{1/n^2} = 1/(n^n)^{1/n^2} = 1/n^{n/n^2} = 1/n^{1/n} \to 1$, so $R = 1$.

(b) Here $|c_n|^{1/n} = 1/(n^n)^{1/n} = 1/n \to 0$ so $R = 1/0 = \infty$.

(9) [10] **Derive** *the formulas for* $\cos(z+w)$ *and* $\sin(z+w)$ *in terms of* $\cos z$, $\cos w$, $\sin z$, $\sin w$ *from the corresponding formula for the exponential function.*

We can show that $4\cos(z+w) = 2\cos z \cdot 2\cos w - 2\sin z \cdot 2\sin w$. Starting on the right, using the *definitions* of sine and cosine, we have
$2\cos z \cdot 2\cos w - 2\sin z \cdot 2\sin w = (e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) - (-i)^2(e^{iz} - e^{-iz})(e^{iw} - e^{-iw})$
$= (e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + (e^{iz} - e^{-iz})(e^{iw} - e^{-iw})$. By inspection the "cross" terms $\pm e^{\pm i(z-w)}$ cancel, leaving
$2e^{i(z+w)} + 2e^{-i(z+w)} = 4\cos(z+w)$.

The other formula may be written $4\sin(z+w) = (?)2\sin z \cdot 2\cos w + 2\cos z \cdot 2\sin w$
$= -i(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) - i(e^{iz} + e^{-iz})(e^{iw} - e^{-iw}) = -i[(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) + (e^{iz} + e^{-iz})(e^{iw} - e^{-iw})]$.
By inspection the "cross" terms $\pm e^{\pm i(z-w)}$ cancel again, leaving $-i[2e^{i(z+w)} - 2e^{-i(z+w)}] = 4\sin(z+w)$.