

Note: Special Problems are like Term Papers. They must be well and neatly written, or typed, on standard size paper. Paper torn from spiral notebooks will not be accepted! Margins of one inch at least on all four edges and legibility are required. Special Problems will to some extent be scored competitively. The use of lined paper and writing on both sides of a page are quite acceptable.

Assignment 13 & last, Book Problems: Due Dec 8

Chapter 3, # 24de; Chapter 4, # 1, 8, 20.

Optional Special problem: Due Dec 12. No late papers accepted!

Suppose that, for every $n \in \mathbb{Z}^+$, we have a complex series $\sum_{k=1}^{\infty} a_{nk}$ and also a convergent series $\sum_{k=1}^{\infty} a_k$, with positive terms. Assume that, for all n and k in \mathbb{Z}^+ , $|a_{nk}| \leq a_k$. [2] Prove that each series $\sum_{k=1}^{\infty} a_{nk}$ converges absolutely. Next, suppose that $b_k := \lim_{n \rightarrow \infty} a_{nk}$ exists for each $k \in \mathbb{Z}^+$. [3] Prove that $\sum_{k=1}^{\infty} b_k$ converges absolutely. [10] Prove that $\sum_{k=1}^{\infty} a_{nk} \rightarrow \sum_{k=1}^{\infty} b_k$ as $n \rightarrow \infty$.

Special Problem 7: Due Dec 10

Let $f_m : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as follows: $f(0, 0) := 0$, and $f(x, y) = \frac{xy^m}{x^2 + y^{2m}}$ if $(x, y) \neq (0, 0)$, where m is a positive integer. Note that f_m is defined for all positive integers! Prove that each f_m is continuous at (x, y) if $(x, y) \neq (0, 0)$, and that no f_m is continuous at $(0, 0)$.

Special Problem 6: Due Dec 8 (corrected!)

Prove that, if X is a metric space and $f : X \rightarrow \mathbb{R}$, then the set of points of continuity of f is a G_δ , namely a subset of X that can be expressed as the intersection of an at most countable collection of open sets.

Assignment 12, Book Problems: Due Dec 1

Chapter 3, # 24bc; Chapter 4, # 6.

Assignment 11, Book Problems: Due Nov 24

Chapter 3, # 10, 11cd, 12b, 24a (see note just below).

Note: *Equivalence relation* is defined on page 25: **2.3**. A *relation* on a set S is defined to be a set of ordered pairs contained in $S \times S$. An equivalence relation is thus a special kind of relation. In Exercise 24, the set S is the collection of all Cauchy sequences of points in X . From 24a to the notation of **2.3**: if $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences then $\{p_n\} \sim \{q_n\}$ means $\lim_{n \rightarrow \infty} d(p_n, q_n) = 0$. Exercise 24a asks you to prove that the three properties in **2.3** are true for its way of defining *equivalent*, i.e., \sim .

Assignment 10, Book Problems: Due Nov 17

Chapter 3, # 7, 8, 11ab, 12a.

Special Problem 5: Due Nov 14

Chapter 3, # 14e

Assignment 9, Book Problems: Due Nov 10

Chapter 3, # 16abc, 20.

Special Problem 4: Due Nov 5 (see **3.16**)

Prove that, if $\{s_n\}$ is a sequence of real numbers, then $\limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sup_{m \geq n} s_m$ (this can be an *extended* real).

Assignment 8, Book Problems: Due Nov 3

Chapter 2, # 28; Chapter 3, # 4, 5, 14cd.

Special Problem 3: Due Oct 29

Chapter 2, # 27.

Assignment 7, Book Problems: Due Oct 27

Chapter 2, # 26; Chapter 3, # 1, 2, 3, 14ab.

Assignment 6, Book Problems: Due Oct 20

Chapter 2, # 15, 24, 25, 29.

Assignment 5, Book Problems: Due Oct 13

Chapter 2, # 13, 14, 22, 23.

Assignment 4, Book Problems: Due Oct 8

Chapter 1, # 7efg; Chapter 2, # 6(the rest), 9abc.

Assignment 3, Book Problems: Due Sept 29

Chapter 1, # 7cd; Chapter 2, # 6(first “Prove that”), 7, 8.

Special Problem 2: Due Sept 26

The statement of the Problem comes after some definitions and examples! look for **Problem** on the next page...

(1) **Definition:** An inner-product space with complex scalars, \mathbb{C} , is a vector space V with complex scalars, and a complex-valued function $\langle v, w \rangle$, called the inner product, defined on $V \times V$, that has the following properties:

- (a) For all $v \in V$, $\langle v, v \rangle \geq 0$.
- (b) If $\langle v, v \rangle = 0$ then $v = 0$.
- (c) For all v and w in V , $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
- (d) For all v_1, v_2 and w in V , $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$.
- (e) For all v, w in V , and all scalars a , $\langle av, w \rangle = a \langle v, w \rangle$.

In case the scalars are real, the axioms are the same, except that $\langle v, w \rangle$ is assumed to be real-valued, so the complex conjugation is dropped in (c): $\langle v, w \rangle = \langle w, v \rangle$.

The “dot product” in Euclidean space is the basic example of an inner product (the scalars are real in that case...). Note that $\operatorname{Re} \langle v, w \rangle$ is an inner product on the real vector space obtained by restricting scalar multiplication to the real numbers. This will be important later.

(2) **Examples** include \mathbb{C} itself, with

$$\langle z, w \rangle = z\bar{w};$$

$C([0, 1])$, the complex-valued continuous functions on $[0, 1]$ with

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)} dx;$$

and $L^2(\mathbb{R}^n)$, with

$$\langle f, g \rangle := \int f(x)\overline{g(x)} dx.$$

Other examples include the “usual” finite-dimensional spaces of vectors, such as 3-space (\mathbb{R}^3), with the “dot product” playing the rôle of inner product (in this example the scalars are real), \mathbb{R}^n (or n -space), consisting of column vectors (x_1, x_2, \dots, x_n) or row vectors $(x_1 x_2 \dots x_n)$ with inner product (still a dot product, with real scalars)

$$\langle x, y \rangle = x \bullet y = \sum_{k=1}^n x_k y_k.$$

We can also work with the similar vector spaces \mathbb{C}^n whose vectors z have complex numbers as coordinates, with inner product

$$\langle z, w \rangle = z \bullet w = \sum_{k=1}^n z_k \overline{w_k}.$$

In passing, we notice that these last inner products can be expressed as products of matrices. We can write (in the context of \mathbb{R}^n consisting of column vectors)

$$\langle x, y \rangle = x^T y = \sum_{k=1}^n x_k y_k,$$

where x^T is the $1 \times n$ matrix obtained by turning the column into a row. The matrix product $x^T y$ is well-defined, and its result is a 1×1 matrix that we treat as a scalar. Similarly, in \mathbb{C}^n ,

$$\langle z, w \rangle = z^T \bar{w} = \sum_{k=1}^n z_k \bar{w}_k.$$

These things are assumed known by the user of standard computer “packages” that do the details of linear-algebra computations for us.

We next define the length of a vector, and call it the “norm” of the vector. The distance between two vectors v and w is then the length of (the norm of) the vector $v - w$. In the context of vector spaces, “norm” is a technical term that picks out the essential features of the concept of length.

(3) **Definition:** A norm on a vector space V is a real-valued function defined for each vector v in V , usually denoted $\|v\|$, with properties (i) – (iii) below, assumed true for all vectors v and w in V and for all scalars c (usually complex numbers for us, but the scalars are often real numbers):

- (i) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$ (the zero vector);
- (ii) $\|cv\| = |c|\|v\|$;
- (iii) $\|v + w\| \leq \|v\| + \|w\|$, the triangle inequality.

Definition (3) was “general;” the next one is specific to inner product spaces.

(4) **Definition:** The norm of an element v in an inner product space is denoted $\|v\|$, and is given by taking the non-negative square root of $\|v\|^2 = \langle v, v \rangle$. That is, $\|v\| = \sqrt{\langle v, v \rangle}$.

Simply calling this a “norm” does not make it one! To prove this is a norm, we’ll use the very important *Schwarz inequality* in the proof of the triangle inequality.

(5) **Theorem (The Schwarz Inequality):** In an inner product space V , for all vectors v, w ,

$$|\langle v, w \rangle| \leq \|v\| \|w\|,$$

and equality holds if and only if one of v and w is a multiple of the other.

Problem: Two parts

(A): Prove: When (c) is combined with (d) and (e) in turn, we have

(d') For all v_1, v_2 and w in V , $\langle w, v_1 + v_2 \rangle = \langle w, v_1 \rangle + \langle w, v_2 \rangle$.

(e') For all $v, w \in V$, and all scalars a , $\langle v, aw \rangle = \bar{a} \langle v, w \rangle$.

Thus the inner product is linear in the first variable, and *conjugate linear* in the second.

(B): Prove (5), The Schwarz Inequality, in the inner-product space context.

Assignment 2, Book Problems: Due Sept 22

Chapter 1, # 6cd, 7ab; Chapter 2, # 2, 5.

Assignment 1, Book Problems: Due Sept 15

Chapter 1, # 2, 3, 6ab, 8, 10.

Special Problem 1: Due Sept 8

Prove that the rational numbers (the ones you “brought with you”) have the Archimedean property. Be careful that you only use the definition of \mathbb{Q} that we gave in terms of \mathbb{Z} !