The exponential series

For $z \in \mathbb{C}$ the function $\exp(z)$ is defined by the power series

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which converges absolutely, by the Ratio Test, for all $z \in \mathbb{C}$. As an application of the M-test, if $|z| \le A$ the series of functions $z^n/n!$ converges uniformly, so $\exp(z)$ is continuous, as a function of z. Let us show that $\exp(z)$ is differentiable, in the complex sense, at $z = 0$.

Lemma: If $0 < |z| \le 1$ then $\left|\frac{\exp(z)-1}{z} - 1\right| \le (e-2)|z|$. Thus $\lim_{z\to 0} \frac{\exp(z)-1}{z} = 1$.

Proof: We have

$$\frac{\exp(z)-1}{z} - 1 = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} - 1 = \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+2)!} = z\sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}.$$

If $|z| \le 1$, then

$$\left|\sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}\right| \le \sum_{n=0}^{\infty} \frac{1}{(n+2)!} = e - 2.$$

The inequality is now shown, and the rest of the Lemma follows, when we apply the Squeeze Principle.

We can extend the differentiability at zero to differentiability at each z by using the "Law of Exponents" property of the exponential function.

Theorem: *For all complex numbers* z *and* w, $\exp(z+w) = \exp(z)\exp(w)$.

By combining the Theorem and the Lemma we have

Theorem: *For all complex numbers* z,

$$\exp(z) = \frac{d}{dz}\exp(z) = \lim_{h\to 0} \frac{\exp(z+h) - \exp(z)}{h}.$$

Proof:

$$\frac{\exp(z+h)-\exp(z)}{h} = \exp(z)\frac{\exp(h)-1}{h} \to \exp(z), \quad \text{as} \quad h \to 0.$$

Toward proof of the Law of Exponents

We can begin a proof by writing down the series for the exponential of $z + w$ and then use the Binomial Theorem on each term:

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} z^m w^{n-m} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!},$$

in which we use the formula $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ from the Binomial Theorem. We can now write

$$\exp(z+w) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} [m \le n] \frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!}.$$

If we could interchange the summations here without changing the value of the double sum, we would have

$$(*) \qquad \exp(z+w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} [m \le n] \frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} [m \le n]\frac{w^{n-m}}{(n-m)!}\right)\frac{z^m}{m!}.$$

Here the inner sum,

$$\sum_{n=0}^{\infty}[m \le n]\frac{w^{n-m}}{(n-m)!},$$

does not depend on m, and we have

$$\sum_{n=0}^{\infty}[m \le n]\frac{w^{n-m}}{(n-m)!} = \sum_{n=0}^{\infty}\frac{w^n}{(n)!} = \exp(w).$$

Thus in $(*)$ we have

$$\exp(z+w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[m \le n]\frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!} = \sum_{m=0}^{\infty}\exp(w)\frac{z^m}{m!} = \exp(z)\exp(w).$$

This would complete the proof, but we need to know that we can indeed reverse the order of summation! The original double sum was an infinite series of finite sums. The new series is an infinite series of infinite series!

On rearranging series

Theorem: *If* a_{nm} *are non-negative numbers, for* $n \in \mathbb{N}$ *and* $m \in \mathbb{N}$, *then*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}a_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}a_{nm}\right).$$

Remark: One use of this Theorem occurs in checking the hypothesis of the next one. It tells us that if it is more convenient to sum in one order that the other, we can do so without changing the result.

Proof: There is no requirement here that either sum be finite. Let A be a real number that is less than the term on the left, and for each $n \in \mathbb{N}$ define the extended real number $A_n := \sum_{m=0}^{\infty}a_{nm}$. If some $A_{n_o} > A$ then there exists M_{n_o} such that $\sum_{m=0}^{M_{n_o}}a_{n_o m} > A$. In this case, we let $N = n_o$, and we select $M_n = M_{n_o}$ for $0 \le n \le n_o$. Otherwise, $A_n \le A$ for all n but there exists $N \in \mathbb{N}$ such that

$$\sum_{n=0}^{N}A_n = \sum_{n=0}^{N}\left(\sum_{m=0}^{\infty}a_{nm}\right) > A.$$

We set $\epsilon := \displaystyle\sum_{n=0}^{N}A_n - A > 0$, and choose, for each n in the sum, M_n such that $\displaystyle\sum_{m=0}^{M_n}a_{nm} + (\epsilon/N) > A_n$. Then

$$A + \epsilon = \sum_{n=0}^{N}A_n < \sum_{n=0}^{N}\left(\sum_{m=0}^{M_n}a_{nm} + \frac{\epsilon}{N}\right) = \left(\sum_{n=0}^{N}\sum_{m=0}^{M_n}a_{nm}\right) + \epsilon.$$

Therefore, if we put $M := \max(M_0, \ldots, M_N)$, and use asscoiativity and commutativity of addition,

$$A < \sum_{n=0}^{N}\sum_{m=0}^{M_n}a_{nm} \le \sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm} = \sum_{m=0}^{M}\sum_{n=0}^{N}a_{nm} \le \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{nm}.$$

Since A was arbitrary, the series on the left is at most as large as the one on the right. The same agrument, applied on the right side first shows that the reverse inequality holds. The proof is done.

Theorem: *If* a_{nm} *are complex numbers, for* $n \in \mathbb{N}$ *and* $m \in \mathbb{N}$, *and*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}|a_{nm}|\right) < \infty,$$

then the following series converge, and to to same sum:

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{nm}\right).$$

Proof: Let us write $a_{nm} =: x_{nm} + iy_{nm}$. Then $|x_{nm}| \le |a_{nm}|$ and $|y_{nm}| \le |a_{nm}|$ for all n and m. We recall that $x_{nm} = x^+_{nm} - x^-_{nm}$. Therefore since $0 \le x^{\pm}_{nm} \le |x_{nm}|$, we have by Comparison and the Theorem just proved,

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x^+_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} x^+_{nm}\right) \quad\text{and}\quad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x^-_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} x^-_{nm}\right).$$

Subtraction now gives

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} x_{nm}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} x_{nm}\right).$$

The same argument applies to the terms y_{nm}, and we can multiply by i and add termwise on both sides to complete the proof.

Application to the Law of Exponents

We had $a_{nm} = [m \le n]\frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!}$. We need to verify that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left|[m \le n]\frac{w^{n-m}}{(n-m)!}\frac{z^m}{m!}\right| = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[m \le n]\frac{|w|^{n-m}}{(n-m)!}\frac{|z|^m}{m!} < \infty.$$

Here it is convenient to work in the other order, with

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[m \le n]\frac{|w|^{n-m}}{(n-m)!}\frac{|z|^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}[m \le n]\frac{|w|^{n-m}}{(n-m)!}\right)\frac{|z|^m}{m!}.$$

The inner sum (as we saw before) does not depend on m. It is equal to

$$\sum_{n=0}^{\infty}\frac{|w|^n}{(n)!}, \quad \text{and so the double sum of absolute values is} \quad \exp(|w|)\exp(|z|) < \infty.$$

This completes the proof of the Law of Exponents. All the proofs so far are now done. Therefore we will adopt the

New notation for the exponential function $e^z := \exp(z)$,

although the old one will be used if it is convenient to do so.

Some miscellaneous properties of the exponential function

1: $\exp(\overline{z}) = \overline{\exp(z)}$. This follows from examining the series.

2: $e^0 = \exp(0) = 1$. This follows from examining the series.

3: For all $z \in \mathbb{C}$, $e^z = \exp(z) \ne 0$. We have $1 = \exp(0) = \exp(z)\exp(-z)$.

The trigonometric functions

The miscellaneous properties and the Law of Exponents tell us that, on the imaginary axis, the exponential has absolute value one. That is, its values lie on the unit circle. We define the functions $\cos x$ and $\sin x$ to be, respectively, the real and imaginary parts of $\exp(ix)$. Thus

Definition: $\cos x := \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x := \dfrac{e^{ix} - e^{-ix}}{2i}$.

Remark: We can use these definitions when x is complex as well! For the moment, we will only study the case of real x.

By the Chain Rule, or by the same sort of argument we used to show differentiability in the complex sense, we show (using the definitions) that

$$\frac{d}{dx}\cos x = -\sin x, \quad \text{and} \quad \frac{d}{dx}\sin x = \cos x.$$

We need to associate a point on the unit circle with each real x. Conversely, given a point z on the unit circle we want to know all real t such that $z = e^{it}$, if any.

Lemma: *There exists a positive number θ such that $e^{i\theta} = i$.*

Proof: To begin, let's examine $e^i = a + ib := \cos 1 + i \sin 1$. We have

$$e^i = \sum_{n=0}^{\infty} \frac{i^n}{n!} = \sum_{n=0}^{\infty} \left\{ \left(\frac{1}{(4n)!} - \frac{1}{(4n+2)!} \right) + i \left(\frac{1}{(4n+1)!} - \frac{1}{(4n+3)!} \right) \right\}.$$

After simplification we get

$$e^i = \sum_{n=0}^{\infty} \left\{ \left(\frac{16n^2+12n+1}{(4n+2)!} \right) + i \left(\frac{16n^2+20n+5}{(4n+3)!} \right) \right\}.$$

Thus $a > 0$ and $b > 0$. The sum of the first two real parts is smaller than the first imaginary part:

$$\frac{0+0+1}{(2)!} + \frac{16+12+1}{(4+2)!} = \frac{1}{2} + \frac{29}{720} < \frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

the last quantity being the first imaginary part. After that, we have

$$\frac{16(n+1)^2 + 12(n+1) + 1}{(4(n+1)+2)!} \frac{(4n+3)!}{16n^2+20n+5} = \frac{16n^2+44n+29}{16n^2+20n+5} \frac{1}{(4n+6)(4n+5)(4n+4)}.$$

By long division, $\dfrac{16n^2+44n+29}{4n+4} = 4n+7+\dfrac{1}{4n+4}$, so the ratio of "real part $n+1$ over imaginary part n," when $n \geq 1$, is less than

$$\frac{4n+7+1}{16n^2+20n+5} \frac{1}{(4n+6)(4n+5)} < \frac{4n+8}{20n+20} \frac{1}{(4n+6)(4n+5)} < 1.$$

Hence $0 < a < b$. Then $e^{2i} = (a+ib)^2 = a^2 - b^2 + 2iab$, so $\cos 2 < 0 < \cos 1$. Hence there exists $\theta > 0$ such that $\cos\theta = 0$. Since the set of all such θ is closed (because $\cos 0 = 1$), we choose the least such θ, and call it $\pi/2$.

Let us show that $1 < \pi/2 < 2$. The second inequality is immediate. To show the first, we note that, if $0 < x \leq 1$, then

$$e^{ix} = \sum_{n=0}^{\infty} \left\{ \frac{x^{4n}}{(4n)!} \left(1 - \frac{x^2}{(4n+2)(4n+1)} \right) + i \frac{x^{4n+1}}{(4n+1)!} \left(1 - \frac{x^2}{(4n+3)(4n+2)} \right) \right\}.$$

Each real part and each imaginary part is positive, so $\cos x$ and $\sin x$ are both positive on $[0, 1]$. Thus $\pi/2 > 1$. Moreover, $\cos x > 0$ on $[0, \pi/2)$, so $\sin(\pi/2) > 0$. Hence $\sin(\pi/2) = 1$ because $\sin^2(\pi/2) = 1$.

We have shown that $e^{i\pi/2} = i$.

Corollary: *Let $\pi/2$ denote the smallest positive number θ such that $\cos\theta = 0$. Then*

$$e^{i\pi/2} = i, \; e^{i\pi} = -1, \; e^{i3\pi/2} = -i, \; e^{2\pi i} = 1.$$

Moreover, $\cos x$ decreases strictly, and $\sin x$ increases strictly on $[0, \pi/2]$, and each has image $[0, 1]$ there.

As an application of the foregoing we can now prove the following Theorem.

Theorem: *The function* $U(\theta) := e^{i\theta}$ *is periodic of period* 2π*, and maps the interval* $[0,\, 2\pi)$ *onto the unit circle in the complex plane in a one-to-one manner.*

Proof: We know from the Corollary that U is one-to-one on $[0,\, \pi/2]$. Let us show that the image of $[0,\, \pi/2]$ with respect to U is the portion of the unit circle in the closed first quadrant. We know already that $U(0) = 1$ and that $U(\pi/2) = i$. Suppose that $z = x + iy$ and that x and y are both positive, with $x^2 + y^2 = 1$. Then there exists $\theta \in (0,\, \pi/2)$ such that $\cos\theta = x$. Then $0 < \sin\theta = y$ since $\sin^2\theta = 1 - x^2 = y^2$ and $y > 0$.

Next we consider the behavior of U on $[\pi/2,\, \pi]$. If $\pi/2 < \theta < \pi$, we set $\eta := \pi - \theta$. Then

$$e^{i\theta} = e^{i\pi - \eta} = -e^{-i\eta} = -\overline{e^{i\eta}} = -\overline{(\cos\eta + i\sin\eta)} = -(\cos\eta - i\sin\eta) = -\cos\eta + i\sin\eta$$

which shows that the quarter-circle in the closed first quadrant is mapped onto the quarter-circle in the closed second quadrant in a one-to-one manner. We make use of the invertible linear mapping whose matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the fact that the composition of one-to-one and onto mappings is one-to-one and onto. In particular, to be sure that the map is onto the quarter-circle in the closed second quadrant, we can find the inverse image of a point thereon, which will be a point on the quarter-circle in the closed first quadrant, hence find an appropriate $\eta \in (0,\, \pi/2)$, then set $\theta = \pi - \eta$.

This shows that the function U maps the interval $[0,\, \pi]$ onto the part of the unit circle in the closed upper half plane.

Finally, the linear mapping given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ maps the part of the unit circle in the open upper half plane onto the part of the unit circle in the open lower half plane in a one-to-one manner. It follows that the mapping $U : [0,\, 2\pi) \to \mathbb{C}$ is a one-to-one and onto mapping of that interval to the unit circle.

More on the exponential function

The exponential function is periodic of period $2\pi i$. Thus its behavior can be described by restricting our attention to a certain strip.

Theorem: *Let* S *denote the strip* $\{z \in \mathbb{C} : z = x + iy \text{ and } 0 \le y < 2\pi\}$*. Then the mapping* $\exp : S \to \mathbb{C}$ *given by* $w = \exp(z)$ *is one-to-one, and takes on all non-zero values of* w*.*

Proof: To prepare for the proof we need to discuss the behavior of the exponential function on the real axis. It follows from examination of the series that e^x is real when x is real. The same examination yields the result that $e^x \ge 1$ for $x \ge 0$. Then the identity $1 = e^x e^{-x}$ shows that $0 < e^x \le 1$ for $x \le 0$. Then the derivative of e^x being e^x shows that e^x is strictly increasing on $\mathbb{R}$. Since $e^1 > 1$, the numbers e^n tend to $+\infty$ as $n \to +\infty$, and tend to 0 as $n \to -\infty$. By the Intermediate Value Theorem, the function e^x takes on all positive values.

The absolute value of e^z is e^x, where x is the real part of z.

Now suppose that $e^{z_1} = e^{z_2}$, where z_1 and z_2 are in S. Then, using real and imaginary parts, we have

$$e^{x_1 - x_2} = e^{i(y_2 - y_1)}.$$

Since $e^{x_1 - x_2}$ is real and positive, and since $|y_1 - y_2| < 2\pi$ we must have $y_1 - y_2 = 0$, and hence $x_1 - x_2 = 0$.

The equation $e^z e^{-z} = 1$ shows that zero cannot be a value.

Suppose that $w \ne 0$. Then, as we have seen, $|w| = e^x$ for some real x. Then $w/|w|$ is on the unit circle, so there is $y \in [0,\, 2\pi)$ such that $w/|w| = e^{iy}$. Then $e^{x+iy} = |w|(w/|w|) = w$.

The lines in S that are parallel to the real axis are mapped onto rays emanating from, but not including the origin.