Integer powers and their inverses

We will make free use of Theorem 1.21 in Rudin's book, together with the rules for *integer* exponents: for positive x and y and for integers m and n,

$$(ER) \qquad x^m x^n = x^{m+n};\ (x^m)^n = x^{mn};\ x^{-n} = 1/x^n;\ x^0 = 1;\ 1^n = 1;\ (xy)^n = x^n y^n.$$

Theorem 1.21 gives us existence and notation for the inverses of the functions x^n when n is a positive integer. We also need the Corollary to 1.21: $(xy)^{1/n} = x^{1/n}y^{1/n}$. When n is a positive integer, the rule $x^{-n} = 1/x^n$ gives us the formula for the inverse of a negative power: $x^{1/(-n)} = x^{-1/n} = 1/x^{1/n}$. Thus the non-zero integer powers of positive x and their inverses are defined. We will take the rules for *integer* exponents in (ER) for granted. In particular, each non-zero integer power and its inverse is a continuous strictly monotone mapping of $\mathbb{R}^+$ onto itself.

Developing x^r when $r \in \mathbb{Q}$, and $x > 0$.

We could proceed as in Chapter 1, Problem 6 to develop the rational powers and arbitary real powers. We will take a slightly different approach that (hopefully) will be less tedious than that. But it will use some simple properties of groups. I hope that the introduction to groups that follows will seem familiar, and that you will agree that using the facts about the subgroup generated by a commutative subset makes intuition come alive. If you're already familiar with groups, please read the material anyway, because it develops the particular group we will use.

To begin we define $x^0 = 1$. Then all the rules in (ER) hold when $n = 0$.

In what follows we will concentrate first on *non-zero* rational powers. Our initial goal is to extend the rule $(x^m)^n = x^{mn}$ to include non-zero rational exponents: $(x^r)^s = x^{rs}$. We begin by defining "group."

A *group* consists of a set $\mathcal{G}$ (ours will be the set of continuous strictly monotone mappings of $\mathbb{R}^+$ onto itself) and a binary operation on $\mathcal{G}$, – written the way multiplication is, by juxtaposition. That is, $(x, y) \to xy$. (*Our* binary operation will be composition of functions: $(f, g) \to f \circ g$, but in this discussion we will use juxtaposition! So $x^2x^3 = (x^3)^2 = x^6$, not x^5!) Some essential assumptions about this operation complete the definition:

(1) The binary operation is *associative,* namely, for all x, y, z in $\mathcal{G}$, we have $x(yz) = (xy)z$.
(2) There exists in $\mathcal{G}$ an element e called an *identity* with the property that for all $x \in \mathcal{G}$, $ex = x = xe$.
(3) For each $e \in \mathcal{G}$ that satisfies (2), and for each $x \in \mathcal{G}$, there exists $y \in \mathcal{G}$ such that $xy = e = yx$. The element y is called an *inverse* of x.

The utility of (1) is that we can dispense with parentheses and write xyz even though we "really" can only operate on *pairs* of elements of $\mathcal{G}$.

In a group, the binary operation is not necessarily commutative! However, if it *is* commutative, the group is called a *commutative group.* We will also call $\mathcal{S} \subseteq \mathcal{G}$ *commutative* if $s_1s_2 = s_2s_1$ whenever s_1 and s_2 belong to $\mathcal{S}$.

There are other properties of the binary operation that can be derived from the defining ones: there is only one element of $\mathcal{G}$ that satisfies (2); inverses are unique, and can then be denoted x^{-1}. Proofs of these are tricky. They begin with the idea: assume two elements have the property, then show they must be the same. Other properties, easier to prove, are: $(x^{-1})^{-1} = x$, $(xy)^{-1} = y^{-1}x^{-1}$ and *cancellation:* if $xy = xz$ (or $yx = zx$) then $y = z$.

In our group $\mathcal{G}$ the identity element is the identity function: $e(x) = x$. The inverse of $p_n(x) := x^n$ is its inverse function $p_n^{-1}(x) := x^{1/n}$. In this context, the inverse of x^n is not its reciprocal! You can probably verify that our class $\mathcal{G}$ *is* a group *"in you head."*

We will let $\mathcal{PZ}$ denote the subset of our $\mathcal{G}$ that consists of the functions $p_n(x) := x^n$, where n is a non-zero integer. Note that this includes negative powers. We will let $\mathcal{PZ}^{-1}$ denote the set of inverses, namely $p_n^{-1}(x) := x^{1/n}$, where n is a non-zero integer. We will soon find it convenient to write $p_n^{\pm 1}$ to stand for "either p_n or its inverse."

We need some more "group" concepts. A subset $\mathcal{H}$ of a group $\mathcal{G}$ is a *subgroup* of $\mathcal{G}$ if, *using in $\mathcal{H}$ the operation of $\mathcal{G}$, "inherited" from $\mathcal{G}$,* $\mathcal{H}$ is a group.

A subgroup $\mathcal{H}$ of $\mathcal{G}$ is *generated by* a subset $\mathcal{S}$ of $\mathcal{G}$ if $\mathcal{H}$ is the smallest subgroup of $\mathcal{G}$ that contains $\mathcal{S}$. To prove that this exists one shows that the intersection of a family of subgroups of $\mathcal{G}$ is a subgroup of $\mathcal{G}$, and that the family of subgroups of $\mathcal{G}$ that contain $\mathcal{S}$ is nonempty (because it contain $\mathcal{G}$). It needs to be shown also that there is only one such subgroup.

We need to identify the subgroup generated by a set $\mathcal{S}$. let us construct the set

$$H(\mathcal{S}) := \{x_1^{\pm 1} x_2^{\pm 1} \cdots x_n^{\pm 1} : n \text{ is a positive integer, and each (arbitrary!) } x_k \in \mathcal{S}\}.$$

In words, this is the set of all possible products of finitely many elements of $\mathcal{S} \cup \mathcal{S}^{-1}$. The claim is that $H(\mathcal{S})$ is the subgroup of $\mathcal{G}$ generated by $\mathcal{S}$. By construction, $H(\mathcal{S})$ contains $\mathcal{S}$. It is an exercise in reading the definition of a group to show that $H(\mathcal{S})$ is a subgroup of $\mathcal{G}$. Finally one shows that every subgroup of $\mathcal{G}$ that contains $\mathcal{S}$ must contain $H(\mathcal{S})$.

Our interest is in $H(\mathcal{PZ})$. We have one more group-theoretic task: to show that, if the elements of $\mathcal{S}$ commute with each other (i.e., if $s_1 \in \mathcal{S}$ and $s_1 \in \mathcal{S}$ then $s_1 s_2 = s_2 s_1$) then $H(\mathcal{S})$ is commutative.

The proof of this is the main point of this digression into group theory. The rules for integer exponents tell us that

$$p_n p_m (= p_n(p_m(x)) = (x^m)^n = x^{mn} = (x^m)^n = p_m(p_n(x))) = p_m p_n, \quad \text{so that } p_n p_m = p_{nm} \text{ as well!}$$

so our $\mathcal{S} = \mathcal{P}\mathbb{Z}$ is commutative. Now we return to the group situation, and assume that the elements of $\mathcal{S}$ commute with each other. Then the same is true for their inverses, for

$$x^{-1} y^{-1} = (yx)^{-1} = (xy)^{-1} = y^{-1} x^{-1}.$$

Next we need to show that $x^{-1} y = y x^{-1}$ is always true. This is easy once the right idea occurs to us: use the observation that $u = v$ if and only if $u^{-1} v = e$. Now we let $u = x^{-1} y$ and let $v = y x^{-1}$. Then

$$u^{-1} v = y^{-1} (x^{-1})^{-1} v = y^{-1} x v = y^{-1} x y x^{-1} = y^{-1} y x x^{-1} = ee = e.$$

We have shown that, if $\mathcal{S}$ is commutative, so is $\mathcal{S} \cup \mathcal{S}^{-1}$.

To finish the proof we suppose that $\xi := x_1^{\pm 1} x_2^{\pm 1} \cdots x_n^{\pm 1}$ and $\eta := y_1^{\pm 1} y_2^{\pm 1} \cdots y_m^{\pm 1}$ are in $H(\mathcal{S})$. Then (inductions can be used) in $\xi\eta = x_1^{\pm 1} x_2^{\pm 1} \cdots x_n^{\pm 1} y_1^{\pm 1} y_2^{\pm 1} \cdots y_m^{\pm 1}$ we can, one by one, "move" each $y_j^{\pm 1}$ to the left, past all the $x_k^{\pm 1}$, from which it follows that $\xi\eta = \eta\xi$. Hence $H(\mathcal{S})$ is commutative.

Our application is to $\mathcal{P}\mathbb{Q} := H(\mathcal{P}\mathbb{Z})$. We have shown that, for any finite sequence of non-zero integers or reciprocals of non-zero integers, the order of composition can be rearranged without affecting the composition. We can also remove any occurences of the *pair* consisting of p_n and p_n^{-1}, no matter where they occur in our finite sequence. This allows us to define a one-to-one correspondence between the elements of $\mathcal{P}\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$.

Somewhat as Rudin does in Chapter 1 # 6, we begin with a non-zero rational number $r = m/n$. Here, m and n are non-zero integers. Then we define $P_r(x) := p_n^{-1} p_m = (x^m)^{1/n} (= (x^{1/n})^m = p_m p_n^{-1}$ by what we have proved). To check your understanding, use what we have done to show that if $m/n = m'/n'$ then $P_{m/n} = P_{m'/n'}$. This shows that P_r is well-defined by the rational number r and not by the particular fraction that is chosen to represent r.

This gives us a mapping $h : \mathbb{Q} \setminus \{0\} \to \mathcal{P}\mathbb{Q} = H(\mathcal{P}\mathbb{Z})$. That is, $h(r) := P_r$. To see that h is one-to-one, suppose that $P_r = P_s$. Say $r = m/n$, $s = j/k$. Since we are given that $P_r = P_s$, $p_m p_n^{-1} p_k p_j^{-1} = e$, or $p_{mk} = p_{nj}$. That is, $x^{nj} = x^{mk}$. There are several ways we can show that therefore $nj = mk$. One way is to differentiate and then set $x = 1$. Thus $r = n/m = j/k = s$. Hence the mapping $h(r)$ is one-to-one. Let us show that h is onto. We have seen that the "general" element of $\mathcal{P}\mathbb{Q}$ can be written $f = p_{n_1}^{\pm 1} \cdots p_{n_K}^{\pm 1}$, where each n_j is a non-zero integer. Since $\mathcal{P}\mathbb{Q}$ is commutative, we may assume that no pair of subscripts appears with opposite exponents $+1$ and -1, for these combine to give e. Again by commutativity, we can arrange the factors so that the $+1$ exponents are together, followed by the factors with -1 exponents. Since $p_n^{-1} p_m^{-1} = p_{mn}^{-1}$ and the same rule applies to the $+1$ exponents, we finally have $f = p_M p_N^{-1}$, where M is the product of the subscripts with exponents $+1$ and N is the product of the subscripts with exponents -1. Thus, $f = P_{M/N} = h(M/N)$.

As an example of using our results about $\mathcal{P}\mathbb{Q}$, we have

$$p_n^{-1} p_m p_\ell^{-1} p_k = p_n^{-1} p_\ell^{-1} p_m p_k = p_{n\ell}^{-1} p_{mk}.$$

In our usual way of writing this, and in the context of Chapter 1 # 6, we have shown that

$$(x^{k/\ell})^{m/n} = (((x^k)^{1/\ell})^m)^{1/n} = (x^{mk})^{1/(\ell n)} = P_{mk/\ell n}(x) = x^{mk/\ell n}.$$

This also shows that (with $r = m/n$ and $s = k/\ell$)

$$P_r(P_s(x)) = P_{sr}(x).$$

In our usual way of writing, this is $(x^s)^r = x^{sr}$, the rule in (ER) that we wanted to prove.

We can next study a *new* group of functions on $\mathbb{R}^+$, *all* the rational powers of positive numbers x, including x^0. In this new group, the binary operation is pointwise multiplication, the identity element is $1 = x^0$, and now the inverse *is* reciprocation. You should check that this *is* a group! This group is a subgroup of the (multiplicative) group of all positive functions on $\mathbb{R}^+$, which is commutative to start with. It is in the context of this group that we can fairly easily prove all the other rules in (ER).

Various properties of x^r, when $x > 0$ and $r \in \mathbb{Q}$.

To verify that all this is usable, please verify all the exponent rules in (ER) for rational exponents. We will next differentiate powers x^r and derive their monotonicity properties.

The derivative of x^r.

Let us write $\xi := x^{1/n}$ and $\eta := y^{1/n}$. Then in the difference quotient we have (influenced by Chapter 1 # 7?)

$$\frac{x^r - y^r}{x - y} = \frac{\xi^m - \eta^m}{\xi^n - \eta^n} = \frac{\xi^{m-1} + \cdots + \eta^{m-1}}{\xi^{n-1} + \cdots + \eta^{n-1}} \to \frac{m\xi^{m-1}}{n\xi^{n-1}} = \frac{m}{n}\xi^{m-n} = \frac{m}{n}\, x^{\frac{m}{n}-1},$$

the usual Calculus formula, as $\eta \to \xi$. We used the continuity result in 1.21.

Monotonicity properties

If $r \in \mathbb{Q}$, and $x > 0$ and $r > 0$, then x^r is strictly increasing.

This can be proved using the formula for the derivative of x^r and the theorem that relates "positive derivative" and "strictly increasing."

If $r \in \mathbb{Q}$, and $x > 0$ and $r < 0$, then x^r is strictly decreasing.

This can be proved similarly, or by using an argument involving reciprocals.

If $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ and $r < s$, then for all $x > 1$, $x^r < x^s$.

We may use the rules for exponents now when the exponents are rational. Therefore $1 = 1^{s-r} < x^{s-r}$ by monotonicity, so $x^r = 1 \times x^r < x^r \times x^{s-r} = x^s$.

In a similar way, we can prove the following property:

If $r \in \mathbb{Q}$, $s \in \mathbb{Q}$ and $r < s$, then for all $x \in \mathbb{R}^+$, $0 < x < 1 \Rightarrow x^r > x^s$.

Four important limits

If $r \in \mathbb{Q}$, and $r > 0$, then $\lim_{x\to\infty} x^r = \infty$ and $\lim_{x\downarrow 0} x^r = 0$, but if $r < 0$, then $\lim_{x\to\infty} x^r = 0$ and $\lim_{x\downarrow 0} x^r = \infty$.
Proof: We can prove this when $r > 0$, then use limit theorems. To show that $\lim_{x\to\infty} x^r = \infty$, we let $R > 0$ be given, no matter how large. Then $r = m/n$ for some positive integers m and n. Let $x \geq R^{n/m}$. Then by substitution and our monotonicity properties, $x^r = x^{m/n} \geq (R^{n/m})^{m/n} = R$. To show the other limit, we let $\epsilon > 0$ be given, no matter how small and use a similar idea.

A useful, but technical, double inequality: toward arbitrary real exponents

We will need an inequality involving rational powers, when x is in an interval that is bounded away from zero and bounded from above. It is convenient to use intervals of the form $[1/a, a]$, where $a > 1$.

If $a \in \mathbb{R}^+$, *and* $a > 1$, *then for all* $x \in \mathbb{R}^+$, *and for all* $r \in \mathbb{Q}$, $1/a \le x \le a \Rightarrow a^{-|r|} \le x^r \le a^{|r|}$.

The proof is tedious, involving cases. If $1 < x \le a$, then $\left(\frac{1}{a}\right)^r \le \left(\frac{1}{x}\right)^r < 1 < x^r \le a^r$, or

$$a^{-r} \le x^{-r} < 1 < x^r \le a^r; \text{ in other "words," } a^{-r} \le x^{\pm r} \le a^r.$$

If $1/a \le x < 1$, then $\left(\frac{1}{a}\right)^r \le x^r < 1 < \left(\frac{1}{x}\right)^r \le a^r$, or similar to went before,

$$a^{-r} \le x^r < 1 < x^{-r} \le a^r, \text{ that is, } a^{-r} \le x^{\pm r} \le a^r.$$

When we let r be an arbitrary rational number (including $r = 0$), the inequality that we had before, for $r > 0$, $a^{-r} \le x^{\pm r} \le a^r$ has to be re-written

$$a^{-|r|} \le x^{\pm r} \le a^{|r|} \text{ which gives the desired inequality: } a^{-|r|} \le x^r \le a^{|r|}.$$

A technical inequality for a difference of rational powers

Theorem (Comparison of Powers): *If* r *and* s *are rational numbers,* k *is an integer at least as large as* $\max(|r|, |s|, 1)$, $a > 1$ *and* $a^{-1} \le x \le a$, *then*

$$|x^r - x^s| \le a^{5k}|r - s|.$$

Proof: We can use the rules for exponents and the Mean Value Theorem:

$$x^r - x^s = x^s(x^{r-s} - 1) = x^s(r - s)\xi^{r-s-1}(x - 1),$$

and then by the "double inequality,"

$$|x^r - x^s| \le a^{|s|}|r - s|a^{|r-s-1|}(a - 1) \le a^{5k}|r - s|.$$

This completes the proof.

The Theorem leads directly to a viable way to define arbitrary real powers of a positive real variable x.

Definition and Theorem: *If* $t \in \mathbb{R}$ *and* $\{r_n\}$ *is a sequence of rational numbers that converges to* t, *then for each* $x > 0$, *the sequence* $\{x^{r_n}\}$ *converges to a limit* $P_t(x)$ *that is independent of the particular sequence of rational numbers that converges to* t.

We will then *define* x^t by $x^t := P_t(x)$. This process begs the question "what does it mean to raise x to the power t when t is irrational?" We will be content to show that the rules for exponents are true.

Proof: Suppose $x > 0$ and that $\{r_n\}$ is a sequence of rational numbers that converges to t. Take $a := x + 1$. Since $r_n \to t$ there is a positive integer k such that $|r_n| \le k$ for all n. By the comparison of powers Theorem,

$$(\dagger) \qquad |x^{r_n} - x^{r_m}| \le a^{5k}|r_n - r_m|.$$

Since $\{r_n\}$ is Cauchy so is $\{x^{r_n}\}$. Thus a limit, $\lambda(x, \{r_n\})$ exists. However, if $\{s_n\}$ is also a sequence of rational numbers that converges to t, then

$$|x^{r_n} - x^{s_n}| \le a^{5k'}|r_n - s_n| \to 0,$$

so $\lambda(x, \{r_n\}) = \lambda(x, \{s_n\})$. The limit thus depends only on x and t so we can define $P_t(x) := \lambda(x, \{r_n\})$, where $\{r_n\}$ is *any* sequence of rational numbers that converges to t. In case $t = p/q$ is rational, we can set $r_n \equiv t$ to see that $P_t(x) = x^t = (x^p)^{1/q}$.

The rules of exponents, for arbitrary real exponents

For all $x > 0$, *and for all real numbers* t *and* u,

$$x^{t+u} = x^t x^u.$$

This follows, using the limit theorems.

For all $x > 0$, *and for all real numbers* t *and* u,

$$x^{tu} = (x^t)^u.$$

Proof: This one is not so easy to prove. To begin, we define $a := \max(x, 1/x)$. We choose an integer k so large that $k \geq |t| + 2$ and $k \geq |u| + 2$. Then we choose rational sequences $r_n \to t$ and $s_n \to u$ such that for all n, $k \geq |r_n| + 1$ and $k \geq |s_n| + 1$.

Then by the rules for rational exponents,

$$(x^{r_n})^{s_n} = x^{r_n s_n} \to x^{tu} \text{ as } n \to \infty.$$

The hard part is showing that $(x^{r_n})^{s_n} \to (x^t)^u$ as $n \to \infty$. We will split the difference:

$$\left|(x^{r_n})^{s_n} - (x^t)^u\right| \leq \left|(x^{r_n})^{s_n} - (x^t)^{s_n}\right| + \left|(x^t)^{s_n} - (x^t)^u\right|.$$

By definition, $\left|(x^t)^{s_n} - (x^t)^u\right| = o(1)$. To show that $\left|(x^{r_n})^{s_n} - (x^t)^{s_n}\right| = o(1)$, we use a result we have already proved, that the function y^{s_n} is differentiable, and the Squeeze Principle.

By the MVT,

$$(x^{r_n})^{s_n} - (x^t)^{s_n} = s_n c_n^{s_n - 1}(x^{r_n} - x^t),$$

where c_n is between x^{r_n} and x^t. Therefore $|s_n| \leq k$, $|r_n| \leq k$ and so $x^{r_n} \leq a^k$ and $x^t \leq a^k$. Hence we have (using the double inequality for powers in $[1/a, a]$) $c_n^{s_n - 1} \leq (a^k)^{|s_n - 1|} \leq a^{k(k+1)}$. This gives

$$\left|(x^{r_n})^{s_n} - (x^t)^{s_n}\right| = \left|s_n c_n^{s_n-1}(x^{r_n} - x^t)\right| \leq k a^{k(k+1)} |x^{r_n} - x^t| = o(1),$$

again by definition, as desired.

As an exercise, you should prove the other rules in (ER), for real exponents.

The continuity of x^t

In the Definition of (and Theorem on) real powers x^t (see (†)) we actually proved local uniform convergence of the rational powers, and continuity follows. Here is a direct proof, that gives a little more information. We will borrow from the previous argument to prove that, for all positive x_o, x^t is continuous at x_o. For convenience we make some small adjustments. We will suppose that, for some $a > 1$, $1/a < x_o < a$. We will then work with x in the interval $(1/a, a)$, and show that x^t is uniformly continuous on $(1/a, a)$. We then choose a rational sequence $r_n \to t$ and an integer k as in the previous proof.

We will show that $|x^t - x_o^t| \leq M|x - x_o|$ for some constant M, and for $x \in (1/a, a)$. We will do so by taking the limit that defines the t^{th} power, after getting the inequality to hold for the rational powers r_n. It will be a review of uniform continuity for you to verify that $|x^t - x_o^t| \leq M|x - x_o|$ implies the asserted uniform continuity.

By the Mean Value Theorem we have

$$(*) \qquad x^{r_n} - x_o^{r_n} = r_n \xi_n^{r_n - 1}(x - x_o),$$

where ξ_n is between x and x_o, hence between $1/a$ and a. We can therefore put absolute values in $(*)$ and replace the quantities on the right-hand side that depend on n by constants, using inequalities. This yields

$$\left|x^{r_n} - x_o^{r_n}\right| \leq k a^{|r_n - 1|}|x - x_o| \leq k a^{k+1}|x - x_o|.$$

Now we can take the limit on the left, as $n \to \infty$, which gives, for each $a > 1$, and each integer $k > |t|$,

$$\left|x^t - x_o^t\right| \le k a^{k+1} |x - x_o|, \quad \text{when both of } x \text{ and } x_o \text{ belong to } (1/a,\, a).$$

This essentially completes the proof of continuity, and even shows that x^t is a locally Lipschitz function.

The differentiability of x^t

Instead of using the difference quotient, we will use the "little-oh" version of the definition of differentiability, and use limits again. Given $x_o > 0$, we will use the same choices of a, r_n and k that we used in the proof of continuity. We know, by Taylor's Theorem, that we can write (for a frozen-in-place x)

$$(**) \qquad x^{r_n} = x_o^{r_n} + r_n x_o^{r_n - 1}(x - x_o) + r_n(r_n - 1)c_n^{r_n - 2}(x - x_o)^2/2,$$

where c_n is between x and x_o, hence between $1/a$ and a. As we have done twice before, we can estimate the last term:

$$(* * *) \qquad \left|r_n(r_n - 1)\right| c_n^{r_n - 2}(x - x_o)^2/2 \le k^2 a^{k+2}(x - x_o)^2/2.$$

Since each of the terms x^{r_n}, $x_o^{r_n}$, $r_n x_o^{r_n - 1}(x - x_o)$ in $(**)$ has a limit as $n \to \infty$, so does the last term, $r_n(r_n - 1)c_n^{r_n - 2}(x - x_o)^2/2$. We give the name $E(x - x_o)$ to this limit, and so we have shown that

$$x^t = x_o^t + t x_o^{t-1}(x - x_o) + E(x - x_o).$$

To complete the proof, all we have to do is show that $E(x - x_o) = o(x - x_o)$. Since

$$E(x - x_o) = \lim_{n \to \infty} r_n(r_n - 1)c_n^{r_n - 2}(x - x_o)^2/2,$$

$$\begin{aligned} \left|\frac{E(x - x_o)}{x - x_o}\right| &= \lim_{n \to \infty} \left|r_n(r_n - 1)c_n^{r_n - 2}(x - x_o)/2\right| \\ &\le k^2 a^{k+2}|x - x_o|/2 = o(1), \end{aligned}$$

so the proof is complete, and we have shown that

$$\frac{d}{dx}\, x^t = t\, x^{t-1}, \quad \text{for } \; x > 0 \; \text{ and } \; t \in \mathbb{R}.$$

Once you have proved all the exponent rules for real exponents, you might try proving this directly, without going back to using limits of rational powers.

Suggestion: You are urged to extend the Comparison of Powers Theorem to real exponents.

Next topic: exponentials and logarithms, in which we let the exponent vary and hold the "x" fixed.