MATHEMATICAL LOGIC

This is where we start. Basic mathematical logic consists of statements, truth-values, and logical operations. We will define the operations NOT, AND, OR, IF AND ONLY IF, IF … THEN and EXCLUSIVE OR in terms of truth tables. We will begin to distinguish between defined and undefined terms.

In this chapter, you can learn a new kind of "arithmetic" — a calculus of statements. Except for a few examples, at first I want you to just follow along, and not worry about assigning any *meanings* to the statements we work with. In fact, we will be working only with *names* of statements, for the most part! The only thing that matters here is the truth-values of the statements we discuss, not what the statements mean, so letters will be used as names of statements. Meaningful statements will come later. For now, please, just make it your business to learn how to make compound statements. Please think of a statement name, such as A or B, as a "place-holder" for a **truth-value**: **TRUE** or **FALSE**. Please note that statement names are variables!

STATEMENTS

We will be working with **mathematical statements**. Here are some examples: "$1 < 2$," "$3 > 4$," "Every integer larger than 1 is positive," "Every continuous function on a closed and bounded interval assumes maximum and minimum values." The first one is a **true** statement. It has the truth-value TRUE. The second one is **false**. Its truth-value is FALSE. Both statements refer to specific numbers. The third one refers to many numbers; it is true, and is a more complicated sort of statement than the first two. It mentions **integers**, and **positive**, apparently a property of some of the integers. The fourth statement is a theorem from Calculus. It is about certain functions and certain sets.

In order for a statement to make sense as a mathematical statement, the terms, or words, that it contains must each have a definite meaning. Some of the terms will have been defined using other known terms — these terms are called defined terms, that is, *previously* defined terms. Some terms used will be primitive or fundamental terms, whose meanings are assumed to be known — these are the **undefined terms**. Usually an undefined term is explained at great length, but not by using defined terms, or other undefined terms. What is our first undefined term in this chapter? YES! It is "mathematical statement!" Another one is "mathematical object!" Some other "primitive" (undefined) terms are **zero**, **one**, and **two**. Much of our discourse will consist of explanations, directions, and so on. Such discourse, or sentences, or statements, are usually not mathematical statements. Most of them will be *about* mathematical statements. The mathematical statements themselves will usually be about **sets** or **functions**. In contrast to everyday statements, mathematical statements are required to have a definite truth-value, or, to be either true or false, but never both, and to always have the *same* truth value.

We will build mathematical statements in a standard way, a way that normally will allow us to determine their truth-values. We will be concerned with whether the mathematical statements that we build are true or false. The process of demonstrating that a true mathematical statement has the truth-value TRUE, or that a false mathematical statement has the truth-value FALSE, is called a **proof**. We will be making proofs that some mathematical statements are true. We will be making proofs that some mathematical statements are false. To do this we need to use some mathematical logic. We will study some, and use it constantly. At first we will skim this material to become familiar with it, so you can refer to it later, as the need arises. This is not a course IN logic; it's a course that USES logic. So we won't be as thorough as a course in mathematical logic would be. We will be using logic somewhat informally.

Except for some examples, the preceding paragraph contains no mathematical statements, merely statements about them, in English. Sometimes we use words in mathematical statements in a different way than in usual language, whether everyday or literary. In mathematical usage, we want a word that has a technical meaning to have the same, precise meaning every time it is used. Ambiguity is not allowed. But everyday language thrives on ambiguity. And we use mathematical terminology ambiguously sometimes, especially when we express ideas. How can you tell? At first, ask!

Here are two more undefined terms: TRUE and FALSE. Part of the explanation of 'mathematical statement' is: a mathematical statement must have a definite truth-value. The *definition* of **truth-value** is: one of TRUE and FALSE. Thus, "truth-value" is now a *defined* term. We similarly *define* the terms "true" and "false" as adjectives to apply to statements that have the truth-values TRUE and FALSE, respectively. Notice that a "term" is not a mathematical statement: a "term" does not have a truth-value. A mathematical statement has to assert that something is true in order to have a truth-value. Statements such as "$1 < 2$," or "$3 > 4$" are mathematical statements. We are going to be a bit more flexible, though, and allow our *idea* of "mathematical statement" to include some assertions, for example "$x > y$," that don't have a truth value, but DO have a truth-value when values for x and y are specified. We will simply call these "statements." The difference between "$x > y$" and "$2 > 1$," or "$3 > 4$" is that "$x > y$" refers to unspecified objects x and y. We call these **variables**, and we need rules for their use. These rules will come later. "Variable" is an undefined term that we use to describe some of the symbols that occur in mathematical statements.

One objective of the course is to encourage you to make your own criteria of exactness, of expectations you have of those who address you, mathematically or otherwise, in order that you understand what they mean. Then you can meet those expectations in your own discourse. If someone else doesn't quite fulfill your expectations of exactness, etc., the thing to do is to ask questions, lots of them!

LOGIC OPERATORS and COMPOUND STATEMENTS

Suppose we are given two mathematical statements, named **A** and **B**. We are going to define some *new* mathematical statements that incorporate **A** and **B**. They are called **compound statements**. Their truth values will be determined *solely* by the truth-values of **A** and of **B**.

(1.01) Exercise: What are the possibilities for the truth-values of **A** and of **B**? How many possibilities are there, for their pairs of truth-values? How many possibilities are there, for the triples of truth-values of three statements **A**, **B** and **C**?

☛ truth tables

You might want to skip to the next subsection, and come back to this one later, if at all, for it deals only with the nomenclature and purpose of truth tables

The way we will define compound statements is to list all the possible combinations of the truth-values (abbreviated **T** and **F**) of the simple statements (that are being combined into a compound statement) in a table, called a **truth table**. The name of each statement is at the top of a (vertical) **column** of the table. The tops of the columns form a (horizontal) **row** of the table, and we will refer to that row as the "zero$^{\text{th}}$" row of the table so that we can refer to the rows with truth-values by numbers starting with 1. One (or more, if more than one compound statement is being defined) column is reserved, on the far right, for the truth value of the compound statement being defined. The truth values used to define the truth-value of the compound statement appear in one row.

Example:

A	**B**	**??**
T	**T**	**TT?**
T	**F**	**TF?**
F	**T**	**FT?**
F	**F**	**FF?**

The four possible combinations of truth-values for **A** and **B** are shown, one possible combination in each row. Thus, "row 2," or the "second row," contains the symbols **T**, **F**, **TF?**, in that order. This signifies that **A** is true and **B** is false. The symbol **TF?** has to be one of **T** or **F**. The double **??** is a collection of symbols that includes only **A** and/or **B** and some operation symbol. The **TF?** in row 2 will then be replaced by the defined truth-value of the statement signified by **??**. Notice that the only important thing about a statement is its truth-value! We might think of the table as an input-output table: the truth-values in the first two columns of row 2 are the "inputs;" the truth-value in the last column of row 2 is the "output," namely the truth-value *assigned* by the truth-table *to* the compound statement **??** when **A** is false and **B** is true. The Logic operators will all be officially defined by truth-tables!

☛ **the denial of a mathematical statement**

The simplest compound statement is built using just one mathematical statement, say **A**, and the word "**not**." The new statement is called "**the denial of A**." We write "**not A**" for the denial of A. The symbol **~**, the words "**it is not true**," or some other, synonymous construction may also be used. Thus "**not A**," "**~A**," "**A is not true**," mean the same thing. Here, "**not**" is a **logic operator**. The symbol "**~**" is called **the negation sign**. The logic operator "**not**" is called a **unary** logic operator because it operates on one statement. Please make up your own way to deny a mathematical statement. None of the foregoing is our official definition of the denial of A. That distinction is reserved for the **truth table** for **not A**:

(1.02) **Truth Table for: not A**

A	**~A**
T	**F**
F	**T**

The truth-values, abbreviated, of each statement appear in the column below it. This truth table is for two statements, **A** and **~A**. The last column is reserved for the truth-values of the compound statement being defined.

It is very important for you to realize that a compound statement is itself a statement!

A compound statement may have a truth-value different than some or all of the statements that are part of it. The truth-value of **not A** is the opposite of the truth-value of **A**. This last sentence is also a valid definition of **not A**. We will always use truth tables to define logic operators, though. The meaning of **not A** is probably the same as your everyday meaning.

(1.03) Exercise: Suppose that **C** is a false statement. What is the truth-value of the compound statement **~C**? What is the truth-value of the compound statement **~(~C)**? Notice the use of parentheses to make it clear that "**not**" is applied to ONE statement.

☛ **the combination of two statements into one that is true just when *both* are true**

Suppose we have two mathematical statements, say **A** and **B**. We will define a new statement, called by some their **conjunction**, that is true when both **A** and **B** are true, and false otherwise. It is built with the word "**and**," the symbol "$\wedge$," or some other, synonymous construction. The new statement is called "**A and B**." Here, "**and**" is a logic operator, or **logical connective**. It is called a **binary** logic operator because it operates on two statements. Here is its truth table:

(1.04) **Truth Table for: A and B**

A	B	A∧B
T	T	T
T	F	F
F	T	F
F	F	F

The truth-values of each statement appear in the column below it. All 4 possibilities for the pairs of truth-values of **A** and of **B** are shown in the two columns on the left. The (**defined!**) truth-value for **A∧B**, when **A** and **B** have a particular pair of truth-values, is in the same row as the particular pair of truth-values. For example, when **A** is false and **B** is true (second row from the bottom), **A∧B** has the assigned truth-value FALSE. The meaning of **A and B** is probably the same as its everyday meaning.

> **(1.05) Exercise**: Suppose that **C** is a false statement, and **D** is a true statement. What is the truth-value of the compound statement **(~C)∧D**? What is the truth-value of the compound statement **C∧(~D)**? What is the truth-value of the compound statement **~(C∧(~D))**?

> **(1.06) Exercise**: Suppose that the compound statement **C∧(~D)** is true. What are the truth-values of the statements **C** and **D**? You are **deducing** these truth-values.

☛ **the combination of two statements into one that is true when one or both are true**

Suppose we have two mathematical statements, say **A** and **B**. We will define a new statement, called by some their **disjunction**, that is true when one of, or both of **A** and **B** are true, and false otherwise. The new statement is built with the word "**or**," the symbol "$\vee$," or some other, synonymous construction. The new statement is called "**A or B**." Again we have a logic operator: "**or**." It is a binary logical connective. Here is its truth table:

(1.07) **Truth Table for: A or B**

A	**B**	**A$\vee$B**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

The truth-values of each statement appear in the column below it. All 4 possibilities for the truth-values of **A** and of **B** are shown; the truth-value for **A$\vee$B** defined by the table is in the same row as the corresponding truth-values for **A** and for **B**.

(1.08) Exercise: Suppose that **C** is a false statement, and **D** is a true statement. What is the truth-value of the compound statement **(~C)$\vee$D**? What is the truth-value of the compound statement **C$\vee$(~D)**? What is the truth-value of the compound statement **C$\vee$D**? What is the truth-value of the compound statement **~(C$\vee$(~D))**?

The meaning of **A or B** is probably NOT QUITE the same as its everyday meaning! The mathematical-logic meaning is often rendered in everyday language as "and/or." In many everyday contexts, "or" means "one or the other, not both." That is used in logic too, and is known as "exclusive or."

(1.09) Exercise: Please make a truth table for "exclusive or," which I will denote **A $\oplus$ B**:

Truth Table for: "exclusive or"

A	**B**	**A$\oplus$B**

The truth-values of each statement should appear in the column below it. All 4 possibilities for the truth-values of **A** and **B** should be shown; the (defined) truth-value for **A$\oplus$B** should be in the same row.

Maybe the next way of combining two statements should be "**If A, then B**." Instead, let's combine **A** and **B** in such a way that the truth of the new statement would mean that **A** and **B** have the same truth-value, whether TRUE or FALSE.

☛ **the "logic" version of equality is called equivalence, meaning: equal truth-value**

Suppose we have two mathematical statements, say **A** and **B**. We will define a new statement that has truth-value TRUE when both of **A** and **B** are true, *and* when both of **A** and **B** are false. The new compound statement has truth-value FALSE otherwise, that is, when **A** is false, and **B** is true, or vice versa. It is built with the words "**is equivalent to**," the symbol "⇔," the symbol "≡," the words "**if and only if**," the abbreviation "**iff**," or some other, synonymous construction. Thus, "**A is equivalent to B**," "**A ⇔ B**," and "**A if and only if B**" have the same meaning. We have yet another binary logic operator, "⇔." Here is the truth table:

(1.10) **Truth Table for: A if and only if B**

A	B	A⇔B
T	T	T
T	F	F
F	T	F
F	F	T

Important note:

A if and only if B is the way to say, in the language of logic, that (as far as logic is concerned) **A "=" B**.

> **(1.11) Exercise**: Suppose that **C** is a false statement, and **D** is a true statement. What is the truth-value of the compound statement **(~C)⇔D**? What is the truth-value of the compound statement **C⇔(~D)**? What is the truth-value of the compound statement **C⇔D**? What is the truth-value of the compound statement **~(C⇔(~D))**?

Our last way to combine two mathematical statements is **implication**, the one that differs most from everyday usage, and the one probably used more than any other in mathematics: "**If A then B**," "**If A, then B**," "**A implies B**," "**A ⇒ B**" are common notations. Incidentally, **A** is sometimes called the **antecedent**, and **B** the **consequent**. The definition of **If A then B** is a matter of language, *as used in mathematical logic*. So, even if you think it makes no sense, you need to understand this particular construction! The binary logic operator "⇒" is defined, as are the others, by its truth table:

(1.12) **Truth Table for: If A, then B**

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

(1.13) Exercise: Suppose that **C** is a false statement, and **D** is a true statement. What is the truth-value of the compound statement **(~C)⇒D**? What is the truth-value of the compound statement **C⇒(~D)**? What is the truth-value of the compound statement **C⇒D**? What is the truth-value of the compound statement **~(C⇒(~D))**?

(1.14) Exercise: Suppose that the compound statement **C⇒D** is a true statement. In order for **C** to be true, what *must* the truth-value of **D** be? This exercise contains the explanation of why we sometimes say "(the truth of) **D** is a necessary condition for (the truth of) **C**," or "(the truth of) **C** is a necessary condition for (the truth of) **D**," Please notice that we start with the assumption that **C⇒D** is true!

(1.15) Exercise: Suppose that the compound statement **C⇒D** is a true statement. Which truth-value of **C** assures us that **D** is true? This exercise contains the explanation of why we sometimes say "(the truth of) **C** is a sufficient condition for (the truth of) **D**," Please notice that we start with the assumption that **C⇒D** is true!

(1.16) Exercise: Suppose that the compound statement **C⇒D** is false. What are the truth-values of **C** and of **D**?

The main difference with ordinary usage lies in the last two rows of the truth table for **A ⇒ B**. We assign to the compound statement the truth-value TRUE whenever **A** is false, no matter what the truth-value of **B** is! The statement "**A ⇒ B**," when **A** is false, is sometimes called "**vacuously true**," or "**a vacuous statement**." Some students argue with this usage, and I don't disagree, at the everyday level of language, except to point out that those cases don't come up, usually, in everyday situations (but what about in jokes?!). In mathematics, we want mathematical statements to have a truth-value, no matter what the truth-values of statements contained within them may be. In this case it is most useful to assign the value TRUE when **A** is false. Why useful? Here is an example.

Example: "If $a > b$ and $b > c$, then $a > c$." Most people agree that this, interpreted as a statement about numbers, is a true statement. Let us see what this statement means when $a = 1$, $b = 2$, and $c = 0$.

We look at "If $a > b$ and $b > c$, then $a > c$." It has variables a, b and c. When we substitute in the chosen values 1 (for a), 2 (for b) and 0 (for c) we get "If $1 > 2$ and $2 > 0$, then $1 > 0$." First, we need to identify the parts of the statement that play the roles of **A** and **B** in "**If A, then B**," or, in symbols, $\mathbf{A} \Rightarrow \mathbf{B}$. The part "$1 > 2$ and $2 > 0$" will play the role of **A** in $\mathbf{A} \Rightarrow \mathbf{B}$. Notice that **A** is a compound statement, the conjunction of the two statements "$1 > 2$" and "$2 > 0$." The part "$1 > 0$" will play the role of **B** in $\mathbf{A} \Rightarrow \mathbf{B}$.

Now **A**, also known as the **antecedent**, is "$1 > 2$ and $2 > 0$," and is false, and **B**, also known as the **consequent**, namely "$1 > 0$," is true. Therefore, the whole statement, $\mathbf{A} \Rightarrow \mathbf{B}$, is true, according to the truth table, even though **A** is false.

Let us see what this statement means when $a = 1$, $b = 2$, and $c = 3$. The antecedent, "$1 > 2$ and $2 > 3$," is false, and the consequent, "$1 > 3$," is also false. Again, according to the truth table, the whole statement, $\mathbf{A} \Rightarrow \mathbf{B}$, is true. Since we wanted the *whole* statement to be true, this is why we say an implication is true even when the antecedent is false — for in that case, the falsity of the antecedent is not allowed to say anything about the truth-value of the consequent. And we get the benefit that the theorem stays true — we don't have to make special "side" conditions that the antecedent — the **A** statement in **A implies B** — has to be true.

Pardon me for harping on it, but there is another point to be brought out. This time, it will sound more like everyday discourse. The point is, whenever we know, or are given, that the statement **A implies B** is a true statement — that its truth-value is TRUE — and we also know that **A** is true, the truth table says that **B** *must* be true also. This is such an important point that it has a name, **modus ponens** (Latin: method of affirming)[1], and is called a **rule of inference**. It is also an instance of a **principle of deduction**. There is a variant, called **modus tollens** (Latin: method of denying)[1] that says when we know **B** is false, and **A implies B** is true, then **A** must also be false. Please check it out with the help of the truth table!

(1.17) Exercise: Suppose that **C** is a false statement, and **D** is a statement whose truth-value is presently unknown. Suppose that the compound statement $(\sim\mathbf{C})\Rightarrow\mathbf{D}$ is true. What is the truth-value of the statement **D**? What is the truth-value of the statement **D**, if you are given that $(\sim\mathbf{C})\Rightarrow\mathbf{D}$ is false? What is the truth-value of the compound statement $\sim(\mathbf{D}\Rightarrow(\sim\mathbf{D}))$?

We are now done with defining compound statements by means of truth tables. But we are not done with compound statements, or with truth tables! What we have defined is enough to handle anything! Indeed, we have defined more than is actually needed — **and**, **or** and **not** are the only logic operators *needed* to deal with all situations, and some people have even shown how to define a single binary logical connective

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[1]Encyclopædia Britannica, Fifteenth edition.

that can, in turn, be used to define all the others![2] We have defined:

one unary logic operator,

not (symbol: **~**),

four binary logic operators,

and (symbol: $\wedge$),
or (symbol: $\vee$),
if and only if (symbol: $\Leftrightarrow$),
implies (symbol: $\Rightarrow$),

and you have been asked to define a fifth binary operator, **exclusive or** (symbol: $\oplus$). You will be expected to be able to write down the truth tables for any one of these logical connectives.

Important Note:

If we have several statements to put together into a compound statement, we just use these logical connectives, and parentheses, arranged so that we are only connecting two statements at a time with a binary operator, and only negating one statement at a time with the "not" operator. For example, we can't tell what **A**$\wedge$**B**$\vee$**C** means, but we can tell what **(A**$\wedge$**B)**$\vee$**C** and **A**$\wedge$**(B**$\vee$**C)** mean, and their meanings are different!

(1.18) Exercise: Construct the truth tables for each of the compound statements **(A**$\wedge$**B)**$\vee$**C** and **A**$\wedge$(**B**$\vee$**C)**. You can use one "combined" table if you wish. How do they differ?

DENIAL OF COMPOUND STATEMENTS; TAUTOLOGY; CONTRADICTION

We now begin to learn how to use these operators. We will be learning some algebraic sorts of things — indeed, **Boolean algebra** is, in effect what we will study. We begin by showing that the denial of **A and B** is logically the same as **not A or not B**. We will use a truth table.

A	B	A$\wedge$B	~(A$\wedge$B)	(~A)$\vee$(~B)
T	T	T	F	
T	F	F	T	
F	T	F	T	
F	F	F	T	

[2] Mathematical Logic, by Willard Van Orman Quine, revised edition, Harvard U. Press, 1981, pp 45 - 49.

How do we fill in the last column? We look at the partial row 1, under **A** and **B**. We find that both statements are true, so the denial of each one is false. When we connect two FALSEs with **or**, we get FALSE, so we enter F in the last place in the second row of the table. Next, we do the same sort of thing in the next row. There, **T**, **F** get denied to **F**, **T**, and connected by or which gives the result **T** (check, using the truth table for **or** if you need to). Please finish the last column now! You got, if you did it correctly, the result that the last two columns agree underneath their headings. Now, please use the filled-in truth table above to complete the next truth table:

A	B	$\sim(A\wedge B)\Leftrightarrow(\sim A)\vee(\sim B)$
T	T	
T	F	
F	T	
F	F	

If you did it correctly, you found that all the missing entries are **T**. That is, the complicated compound statement $(\sim(A\wedge B))\Leftrightarrow(\sim A)\vee(\sim B)$ is true no matter what the truth-values of the statements that comprise it are. This is known as a **tautology**. If we think of the truth-values in a compound statement as inputs to a process of truth-value calculation, a tautology has output **T** no matter what the inputs are. Please verify that $(\sim A)\vee A$ is a tautology. A compound statement is NOT a tautology if a choice of truth-values can be made for the statements in it, that results in a truth-value **F** for the compound statement. Please notice, in the preceding paragraph, how parentheses were used to make sure that each logic operator had the right number of "operands."

(1.19) Exercise: Define **tautology**.

Just as a tautology is always true, no matter what the truth-values of its component statements (if any) are, a **contradiction** is always false, no matter what the truth-values of its component statements (if any) are.

Example: The statement $A \wedge (\sim A)$ is always false, because **A** and **~A** cannot both be true.

Example: Construct the truth tables for $\sim(A\vee B)$ and $(\sim A)\wedge(\sim B)$ and compare them.
Solution: First, we make a table that displays all the possible combinations of truth-values for **A** and **B**:

A	B
T	T
T	F
F	T
F	F

Then we add a column and put **~(A∨B)** into its top cell:

A	B	~(A∨B)
T	T	
T	F	
F	T	
F	F	

Now the calculations can begin. Row 1 shows that **A** and **B** are both true. Thus, **A∨B** is true, and **~(A∨B)** is false. We'll put **F** in the last cell of row 1. Row 2 shows that now, **A** is true and **B** is false. Then **A∨B** is still true, so **~(A∨B)** is false. We'll put **F** in the last cell of row 2. Row 3 works the same way row 2 did, so we'll put **F** in the last cell of row 3. The last row shows that **A** and **B** are both false, so **A∨B** is false, and thus **~(A∨B)** is true. We'll put **T** in the last cell of row 4, and we get the table

A	B	~(A∨B)
T	T	F
T	F	F
F	T	F
F	F	T

For the next table, the one for **(~A)**∧(**~B)**, we'll work from the one above to get the pairs of truth-values, and just do the calculations, then build the completed table. When the truth-values are **TT**, with the first one always being **A**'s truth-value, and the second one **B**'s, the truth-values for **~A** and **~B** are **FF**, again in **A-B** order. Combining these with ∧ yields the truth-value **F**, so **F** will go into the top empty cell of our table. We can save a little work by noticing that as long as one of **A** and **B** is true, **(~A)**∧(**~B)** must be false, because one of the negated statements in it has to be false, and so the whole conjunction (the fancy name for an "**and**" compound statement) must be false. That means that the next two cells have an **F** too. In the last cell, we'll get a **T**, though, because both of **A** and **B** are false, so the conjunction of their denials must be true. The completed table is:

A	**B**	**(~A)∧(~B)**
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Compare the two tables means to look at the last column, and see how they differ. They do not differ, at least not below their top entries. That is, **~(A∨B)** and **(~A)∧(~B)** have the same truth-values, no matter what the truth-values of **A** and **B** are.

EXERCISES and PROBLEMS

Most, **but not all,** of these exercises amount to "logic equations" that are worth remembering!

(1.20) Exercise: Show, by constructing its truth table, that **(~(A∨B))⇔(~A)∧(~B)** is a tautology. This is an easy exercise; please be sure you *do* it! This is one of **DeMorgan's Laws**.

(1.21) Exercise: Show, by constructing its truth table, that **(~(A∧B))⇔(~A)∨(~B)** is a tautology. This is an easy exercise; please be sure you *do* it! This is the other **DeMorgan's Law**.

(1.22) Exercise: Construct the truth table for the compound statement **((~A)∨B)⇔(A⇒B)**. What does the truth table tell you about the two statements **(~A)∨B** and **A⇒B**?

(1.23) Exercise: Construct the truth table for the compound statement **(B⇒A)⇔(A⇒B)**. What does the truth table tell you about the two statements **B⇒A** and **A⇒B**?

(1.24) Exercise: Construct the truth table for the compound statement **(~B⇒~A)⇔(A⇒B)**. What does the truth table tell you about the two statements **~B⇒~A** and **A⇒B**?
Important: The statement **~B⇒~A** is called the **contrapositive** of the statement **A⇒B**.

Did omitting parentheses in ~B⇒~A confuse you? Officially, it should have been written (~B)⇒(~A), but since we know "~" is unary, ~B and ~A are single statements, so the parentheses are not needed.

(1.25) Exercise: Construct the truth table for the compound statement **((A∨B)∨C)⇔(A∨(B∨C))**. What does the truth table tell you about the two statements **(A∨B)∨C**

and $\mathbf{A}\vee(\mathbf{B}\vee\mathbf{C})$?

(1.26) Problem: In how many ways can the last column of the following table be filled with **T**'s and **F**'s, one per cell? List all of the possibilities. You're building all possible truth tables for compounding two statements!

T	**T**	
T	**F**	
F	**T**	
F	**F**	

(1.27) Problem: This is really several problems! Make one copy of each of the answers to the previous problem, add a row on top to hold **A** and **B** — this makes it look almost like a truth table — then fill in the last entry in your added row with a compound statement made from **A**, **B**, and **logical connectives**, that has that table for its truth table. There can be many correct answers! But the shorter your answer is (by counting characters), the better! Here is an example we've already done:

A	**B**	
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

is one of the tables you get in the previous problem. Having memorized the truth tables for the important logic operators, we recognize that we can put "$\mathbf{A}\Rightarrow\mathbf{B}$" in the empty cell, and get a logic formula for this truth table.

(1.28) Problem: Construct the truth table for the compound statement

$$((\mathbf{A}\Rightarrow\mathbf{B})\wedge(\mathbf{B}\Rightarrow\mathbf{C}))\Rightarrow(\mathbf{A}\Rightarrow\mathbf{C}).$$

One thing you have to do is to figure out how many rows you need to display all the truth-value patterns for **A**, **B** and **C**. The correct number is not four! What does the truth table tell you about the next sentence? If A implies B, and B implies C, then A implies C.

Do the previous **Problem** first — otherwise the following exercises are **Problems**!

(1.29) Exercise: Construct the truth table for the compound statement
$\mathbf{((A\Rightarrow B)\vee(B\Rightarrow C))\Rightarrow(C\Rightarrow A)}$.

(1.30) Exercise: Construct the truth table for the compound statement
$\mathbf{((A\wedge B)\vee(A\wedge C))\Leftrightarrow(A\wedge(B\vee C))}$.

(1.31) Problem: Find a logic formula that is logically equivalent to $\mathbf{(A\vee B)\wedge(A\vee C)}$, but shorter.

(1.32) Problem: How many rows, not counting the top one, are needed to construct the truth table for a compound statement made from N statements?