SETS, VARIABLES AND QUANTIFIERS

Another of our undefined terms is **set**. You are probably familiar with sets, so some of this chapter will be review. But we need to set up notation and terminology. We also need to study **variables** and **quantifiers**. We need some important **fundamental assumptions**.

INTRODUCTION TO SETS

A set consists of **elements** or **members** (of the set). These are also undefined (and synonymous) terms. An element of a set is a **mathematical object**. Sets are often named by capital letters, elements by small letters. For example, if "A" is the name of some particular set, "a" might be used to denote one of its elements. The relationship of the elements of a set to the set that consists of those elements is called **belongs to**, **is an element of**, **is in**, **is a member of**, or some other synonymous construction. The standard symbol for this relation is: $\in$. So, if X is a set, and x one of its elements, we indicate this by writing $x \in X$. We are now going to explicitly assume that "$x \in X$," and its denial, which we will write "$x \notin X$," (so "$x \notin X$," means "$\sim(x \in X$,") are mathematical statements, once a mathematical object x and a set X (also a mathematical object!) have been specified.

A fundamental assumption: ***if X is a set, and x is a mathematical object, then exactly one of the statements "$x \in X$" and "$x \notin X$" is true.***

An important set - our main source of examples for a while — the set N of Natural Numbers

$\mathbf{N} := \{0, 1, 2, 3, \ldots\}$ will denote the set of non-negative integers. For the sake of giving examples, we will use addition and multiplication of non-negative integers, as well as order (less than, less than or equal, and so on), the notion of divisibility, remainders, subtraction when possible, common divisors, and the like. **These things have not been defined yet!** They are being used only to provide examples of new concepts! Please make free use of your knowledge of the non-negative integers in connection with the examples. For example, "two natural numbers are either equal, or one is larger than the other" is to be taken for granted IN EXAMPLES until further notice. We will call the elements of the set **N** **natural numbers**.

Some people use **N** to denote the set $\{1, 2, 3, \ldots\}$ of *positive* integers. For several reasons, including some that lie outside the scope of this course, I find the set $\{0, 1, 2, 3, \ldots\}$ to literally be more "natural," as a basic set of numbers, than the set of positive integers.}

INTRODUCTION TO VARIABLES

A symbol, such as x, when it is used to refer to an unspecified element of a set X, is called a **variable**. "Variable" is an undefined term. In the discussion of set membership, "$x \in X$" has two variables, x and X. Thus, symbols for sets, or names of sets, can be variables. A symbol used to refer to any unspecified mathematical object is a variable too. Also, "$x \in X$" is not really a mathematical statement, because, until specific **values** have been **assigned** to the variables x and X, the statement "$x \in X$" does not have a truth-value. The statement "$x > y$," made in the context of numbers, is not a mathematical statement either, for the same reason. Unrestricted variables in a statement, ones can take any value in some set, are called **free variables**. Variables also occur in set constructions, such as "the set of all x in **N** such that x is even." Here, the variable x takes *all and only* the values, *in* ***N***, that make the statement "x is even" true. These variables are called **bound variables**. The "variable-ness" of x is "used up." In neither case, "$x > y$," or "x is even," is the expression in quotation marks a mathematical statement. But when we choose values for x and for y, such as $x := 3$, $y := 4$, "$x > y$" becomes a mathematical statement that means "$3 > 4$;" "x is even" becomes "3 is even," and both of these are false. The mathematical statements are false, but at least they are mathematical statements. The statement "y is even" becomes the mathematical statement "4 is even," which is a true mathematical statement. For this reason we want to think of expressions like "$x > y$" and "x is even" as *nearly* mathematical statements, so perhaps we can just call them "**statements**." A statement has to have correct *form:* it has to be grammatically correct, lacking only specific values of the free variables that it contains, and it must have a truth-value whenever appropriate values are assigned to its variables.

SUBSETS

A set X is a **subset** of a set Y if every mathematical object that is a member of X is also a member of Y. The phrase "every mathematical object that is a member of X" is a very important one, known as a "quantifier," or "variable binder." We'll study quantifiers in more detail shortly.

Example: Let E denote the set consisting of the even elements of **N**. Now we know what the set E is. This means that, for now, the letter E does not stand for a variable. Then 3, being an odd number, is not a member of E. In other words, "$3 \in E$" is false. Therefore, "$\sim(3 \in E)$" is true. We denote this in symbols by "$3 \notin E$." This is read "three does not belong to E." This could also be expressed in symbols as the compound statement $\sim(3 \in E)$. Note that E is a subset of **N**, so E is a set whose elements are already elements of **N**.

A fundamental assumption: ***if X is a set, and A is a subset of X, then for all x in X, exactly one of the statements "$x \in A$" and "$x \notin A$"*** **is true.**

☛ a rule for treating sets in a special way, to avoid paradoxes

It is very important for us to be careful when making statements about sets, because there are some severe and subtle difficulties that can arise otherwise! We are going to be careful by following a rule about statements involving sets.

RULE: Every mathematical statement that involves a set or sets has to be a statement about a specific set or sets.

We'll fudge a little, by allowing ourselves to make a statement with a variable in it, that looks free, but "really" means a specific, but unspecified set...

☛ some terms and notation involving sets

There are terms that express the relationship of a set to its elements. We say that a set X **contains**, **has**, **possesses**, **owns**, **consists of** its elements. **An important point about set notation**: When we **construct** a set by listing its elements, or by listing enough of them to make it clear what the elements of the set are, we enclose the list in braces, {, and }. Thus, the set consisting of the numbers 1 and 7 is denoted {1, 7}. The set consisting of 2 and 11 is denoted {2, 11}. The set of positive squares can be denoted {1, 4, 9, 16, …}. We can either give sets constructed this way names, or just use the list itself instead of a name.
An important point about sets: a set contains only one copy of each of its elements. Thus, {1, 2, 2, 3} is certainly a list of numbers, enclosed in brackets, but the *set* of numbers contained in the list is {1, 2, 3}: again, a set contains just one copy of each of its elements!

(2.1) ASSIGNMENT NOTATION

An important matter of notation — construction by assignment:
in symbols, we will (continue to) use the computer assignment statement symbol ":=" to mean that the symbol (or word, or phrase) on the left is used to **denote** the mathematical object specified by the symbols on the right of the assignment symbol ("=:" is used when we want to put the denoting

symbol on the right side). The symbol, word or phrase that denotes the mathematical object on the right of ":=" is said to be **defined** by them. Thus, **N** := {0, 1, 2, 3, … } means that the symbol "bold N" will be used to specify the object, in this case a set indicated by the symbols on the right. Usually, when we define a set in this way, we'll use the word **construct**, to distinguish between mathematical terms, which are **defined**, and mathematical objects, which are **constructed**. The distinction is minor, and many people use "define" in both senses. Other examples of set construction: A := {1, 7}, {2, 11} =: B. In the case of **N**, the assignment will be permanent for this course. In the case of A and B, the assignments are temporary. How can you tell which is which? You can usually guess. If you have doubts, ask!

A fundamental rule: *a set cannot be a member of itself!*

This fundamental rule is part of **set theory**. This course is not a course in set theory; it *uses* sets, and does so informally. Set theory is extensive and deep. It dates back to the late 19th century and plays an important role in the part of mathematics called "Foundations." The fundamental rule mentioned above was invented to avoid paradoxes. We need be a little careful about sets, but shouldn't encounter any real problems. You can consult books such as Halmos' *Naive Set Theory* (Undergraduate Texts in Mathematics, Springer-Verlag, 1974) for more information. An especially good source for those wanting to begin a deep study is Chapter 0 of, and the Appendix to, *General Topology*, by J. L. Kelley.

QUANTIFIERS

There are two kinds of **quantifier**, **universal** and **existential**. They will be introduced separately. They are "variable binders:" they allow us to say how to interpret a statement containing variables, as the variable "runs through" a set. Quantifiers turn some statements into mathematical statements. Quantifiers are symbolic "modifiers" of statements that contain variables. Quantifiers can be expressed using symbols or words. Sometimes, quantifiers do not appear explicitly, but are implicitly present, depending on you to realize that they are there!

UNIVERSAL QUANTIFIER

We are used to statements such as: "if $x < y$ and $y < z$, then $x < z$." This statement has the form $P(x, y, z)$. Most of us regard this as a true statement about numbers. So we apparently mean that the statement applies to *all* triples x, y, z of numbers, and that it is a true statement. This means that the variables are not free to take any values, but they must take all possible values!

We make this explicit — because explicitness helps understanding when things get complicated — by using the **universal quantifier**. A statement of the form "**for all** x in X, P(x)" asserts that the statement P(x) is true for each x in X. It might be called a "universal AND." There is a symbol for "For all": $\forall$. Thus the complete version of the statement "if $x < y$ and $y < z$, then $x < z$" is "For all (numbers) x, y, z, if $x < y$ and $y < z$, then $x < z$." This statement has the form $(\forall\ x, y$ and z in $\mathbf{N})P(x, y, z)$. There is an implicit "comma" between the two parts, the one specifying that all triples x, y and z in **N** are being used, and the P(x, y, z) part.
In symbols, this would be written $(\forall x)(\forall y)(\forall z)$(if $x < y$ and $y < z$, then $x < z$), or $(\forall x)(\forall y)(\forall z)(x < y \wedge y < z \Rightarrow x < z)$. It is best to use one copy of the "for all" symbol for each variable. That helps keep confusion down. To be even more precise, we mention the set that each variable belongs to. For example, with $\mathbf{N} = \{0, 1, 2, 3, \dots\}$ as the set containing our numbers, we write

$$(\forall x \in \mathbf{N})(\forall y \in \mathbf{N})(\forall z \in \mathbf{N})(\text{if } x < y \text{ and } y < z, \text{ then } x < z).$$

This is the "complete" form of the "transitive law." This is the form to use when the going gets tough. Notice that there are no longer any free variables. Each of the variables is used up by having to "run through" all the elements of **N**. Here is an example that illustrates the idea of the "universal AND."

Example: Let $X := \{ 1, 2, 3, 4 \}$.
Then $(\forall x \in X)(x > 0)$ means

$$(1 > 0)\wedge(2 > 0)\wedge(3 > 0)\wedge(4 > 0).$$

There is one copy of the statement "$x > 0$" for each of the possible values of the variable x, that is, with x replaced by each of its possible values. The truth-value of the statement $(\forall x \in X)(x > 0)$ is TRUE. The truth-value of $(\forall x \in X)(x > 2)$ is FALSE.
The symbols $(\forall x \in X)$ are referred to as "a universal quantifier," as well as the symbol "$\forall$."

One of the things you will need to learn to figure out is how to start with a statement with no quantifiers, and put in the ones that are implied by context. This is especially important when you work with denials of statements.

We use the term "universal quantifier" to refer to the part of the statement containing "for all" as well as the "for all" itself!

There are many synonyms for "for all x:" **for every x**, **for each x**, **suppose x is arbitrary**, and so on. This will be pointed out, often enough, I hope.

☛ set inclusion, defined again with a universal quantifier

Suppose that A and B are sets. Then **A is contained in B**, denoted **$A \subseteq B$**, if the mathematical statement $(\forall x \in A)(x \in B)$ is true.

EXISTENTIAL QUANTIFIER

Again, let $X := \{ 1, 2, 3, 4 \}$. The truth-value of $\sim(\forall x \in X)(x > 2)$ is TRUE. What does this mean, in terms of the individual statements $(1 > 2), (2 > 2), (3 > 2), (4 > 2)$? Well, if it's not true that all the statements are true, it must be true that at least one of them is false. So, we can say "there exists x in X such that $(x > 2)$ is false." A statement of the form "there exists x in X such that $P(x)$" asserts that the statement $P(x)$ is true for at least one x in X. This quantifier is called the **existential quantifier**. It might be called a "universal OR." There is a symbol for "There exists": $\exists$. In our example, we are saying, "There exists x in X such that $\sim(x > 2)$ is true." We are saying "$\sim(1 > 2) \vee \sim(2 > 2) \vee \sim(3 > 2) \vee \sim(4 > 2)$" is true. Notice that we are not putting parentheses in here — they are not needed, because no matter how they were inserted (as, legalistically, they should be), the result would be the same: the whole statement is true if at least one of its component statements is true. In symbols, we write $(\exists x \in X)(\sim(x > 2))$. Can we simplify $\sim(x > 2)$? Yes, by using a symbol we already know (for some, without knowing we know) for the denial of "$x > 2$:" $x \quad 2$. So we can write: $(\exists x \in X)(x \quad 2)$.

An important point about quantifiers: if they are of the same kind, their order does not matter, BUT the order of quantifiers *does* matter, *very much*, if they are of different kinds!

Example: Once again, let $X := \{ 1, 2, 3, 4 \}$. Then $(\forall x \in X)(\exists y \in X)(x = y)$ is read "for all x in X, there exists y in X such that $x = y$." This statement is a mathematical statement, and it is true, because when we choose $x = 1, 2, 3,$ *and* 4 (in some order) as we must, because the quantifier involving x is universal, we can find y in X, namely $y = x$, such that $x = y$. On the other hand, when we reverse the quantifiers, we have the statement $(\exists y \in X)(\forall x \in X)(x = y)$, which is read "there exists y in X such that for all x in X, $x = y$." This statement is a mathematical statement, and it is false! So the order of the quantifiers can make a big difference!

SET COMPARISON

We know that sets can be compared to each other. We've said that "(the set) A is a subset of (the set) B," if "every element of A is also an element of B" is a true statement. We also say "A **is contained in** B," or "B **contains** A," or "B **is a superset of** A," or, in symbols, "$A \subseteq B$," or "$B \supseteq A$." The relation between sets, $\subseteq$, called **set inclusion**, or **inclusion**, is analogous to for numbers. But the analogy must not be carried too far! We know that, for any two numbers, x and y, that " "x y" or "y x" " is a true statement. This is definitely not true for sets! How can we show that a statement like this is true? It is so vague, so general! Here's how: we make up an example of two sets A and B such that neither statement, "$A \subseteq B$," or "$B \subseteq A$," is true. Here is an example. Let $A = \{1, 2, 3\}$, and let $B = \{0, 1, 2\}$. Then 3 is an element of A, and not an element of B. Therefore, it is not true that every element of A is also an element of B. Thus, the statement "$A \subseteq B$" is false. Similarly, 0 is an element of B, and not an element of A. Therefore, it is not true that every element of B is also an element of A. Thus, the statement "$B \subseteq A$" is false. These two sets are said to be **not comparable**, or **incomparable**.

> **(2.2) Exercise**: Formulate a statement, hopefully a mathematical one, that expresses the notion that two sets A and B are **comparable**, in terms of set inclusion.

When are two sets equal? We can *define* equality for sets, in terms of set inclusion. Two sets A and B are said to be **equal** (as sets) if both mathematical statements "$A \subseteq B$" and "$B \subseteq A$" are true. In other words, "A = B" (for sets) means that A and B have the same elements.

Important note on set equality:

The usual way to verify that, for sets A and B, the mathematical statement "A = B" is true, is to verify that both mathematical statements "$A \subseteq B$" and "$B \subseteq A$" are true. The paradigm is: assume that "a" is an arbitrary element of A (this means that a can be any element of A at all, for example one chosen by someone else). Show, by using known properties of A and B, that "a" also is an element of B. This shows that "$A \subseteq B$" is true. Then, assume that "b" is an arbitrary element of B. Show, by using known properties of A and B, that "b" is an element of A too. This shows that "$B \subseteq A$" is true. Thus, both mathematical statements "$A \subseteq B$" and "$B \subseteq A$" are true. From the definition of equality, we can then say that the sets are equal.

Example: Yet again, let $X := \{ 1, 2, 3, 4 \}$. Let Y denote the set of all positive integers that are less than 5. Then we know X = Y. Here is how we prove that X = Y. Suppose $x \in X$. This means that we let somebody choose x — x has to be completely arbitrary! This means that x is one of the numbers 1, 2, 3 or 4. But then, x is positive and less than 5, no matter which of 1, 2, 3 or 4 x is. Therefore, x meets the criterion for membership in Y, so $x \in Y$. Thus, since

x was any element of X at all, $X \subseteq Y$. On the other hand, suppose that $y \in Y$. This means that y is positive and less than 5. Therefore, y 1, and $y < 5$, so y has to be one of 1, 2, 3 or 4, so $y \in X$. Again, y was any element of Y at all, $Y \subseteq X$. Thus, by the definition of set equality, $X = Y$.

EMPTY SET; SINGLETONS

There is a special symbol reserved for denoting sets with no elements at all: $\varnothing$. A set with no elements is called an **empty set**. For example, the set of all the elements that are in *both* of the sets $A := \{1, 2, 3\}$ and $B := \{4, 5, 6\}$ is an empty set. The "logical" definition of set equality says there is only one empty set. But "psycho-logically," I think it can be useful to think of each set as having its own empty subset.

There is a special name used to describe a set that has exactly one element: a **singleton**. The set consisting of 1, which, you recall is denoted $\{1\}$, is an example of a singleton. **An important point about sets**: 1 and $\{1\}$, as well as $\{\{1\}\}$ are all different! The first, 1, is a number, an element of **N**. The second, $\{1\}$, is the set (a singleton) containing 1. The set $\{1\}$ is a subset of **N**. The last, $\{\{1\}\}$, is also a singleton. Its one element is $\{1\}$, the set containing 1. Thus we see that a set can be a member of another set! A term sometimes used for sets with two elements, such as $\{0, 1\}$, is **doubleton**.

A fundamental assumption: ***if A is a set, then a set exists, whose elements are all of the possible subsets of A. This set contains A and the empty subset of A, and is called the "power set of A," or "the set of all subsets of A," and is denoted, in symbols, by 2^A.***

Examples: Let $A := \{1, 2, 3\}$ (the "Let" is redundant, but we usually don't start sentences with symbols). Then $2^A = \{ \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \varnothing \}$. Notice that "=" was used instead of ":="; 2^A has already been defined. The equal sign means we are going to say just what is in the set. Where does the "2" come into it? Well, A has 3 elements. Please count the number of elements in 2^A. You should get: $8 = 2^3$. There is more to it than that, and we'll come to it later. Let $B := \{4. 5\}$. Then $2^B = \{ \{4, 5\}, \{4\}, \{5\}, \varnothing \}$.

(2.3) Exercise: Let $C := \{1, 2, 3, 4\}$. How many elements do you expect are in 2^C? Write down the set 2^C and check the number of elements in it.

(2.4) Exercise: Write down the set $2^{\varnothing}$ and find the number of elements in it.

(2.5) Exercise: What set is $\{\{1\}\}$ a subset of? What set is $\{\{1\}\}$ an element of? Your answers are to be chosen from $\mathbf{N}$, $2^{\mathbf{N}}$, $2^{2^{\mathbf{N}}}$, $2^{2^{2^{\mathbf{N}}}}$. What do each of these four symbols mean?

A fundamental assumption: ***if A is a set, and P(x) is a statement, that makes sense, about elements, x, of A, then a set exists that consists of <u>a l l</u> the elements x of A such that P(x) is true, and contains no elements z of A such that P(z) is false.***

This fundamental assumption is a mathematical statement about the existence of certain subsets of a given set. We will take it to be a true statement, without proof. A statement that we take to be true without proof we call an **axiom**, or **postulate**.

We use a special notation to denote the sets that this fundamental assumption tells us about. Every time we use this special notation, we are making use of this axiom!

☛ **an important matter of notation**

SET SELECTOR NOTATION:

(2.6) We write $\{ x \in X : P(x) \}$ to denote the subset of X that consists of *all* the elements x of X such that P(x) is true.

We read $\{ x \in X : P(x) \text{ (is true)} \}$ this way: for { we say "The set of all." Then we read the next part, up to the colon. Normally, we say "x belonging to X," or "x in X." Then for the colon, :, we say "such that" and then read P(x), which is a mathematical statement about x. The second brace, }, serves as a period. All at once then, we read: "The set of all x belonging to X such that P(x)."

Examples: Let X be the set $\mathbf{N}$ of natural numbers, $\{0, 1, 2, 3, \dots\}$. Then $\{ x \in \mathbf{N} : x \text{ is even} \} = \{0, 2, 4, 6, \dots\}$; $\{ x \in \mathbf{N} : x < 7 \} = \{0, 1, 2, 3, 4, 5, 6\}$. We read

$\{ x \in \mathbf{N} : x$ is even$\}$ aloud as "The set of all x in $\mathbf{N}$ such that x is even."

(2.7) Exercise: Use set selector notation to define the set of all natural numbers that are divisible by 5.

(2.8) Exercise: Use set selector notation to define the set $\{1, 6, 11, 16, \dots \}$

Set selector notation is compact and convenient. Here is how to use it. There are two ways.
One: If we are given that a particular x, say x_o, is in $\{ x \in X : P(x) \}$, then we know that $P(x_o)$ is a true mathematical statement about x_o. If we are given the set, and we want to say something about it, we often begin by saying, "Suppose x_o belongs to $\{ x \in X : P(x) \}$." This does not necessarily mean that there are any elements in $\{ x \in X : P(x) \}$ at all! What we can truthfully assert, though, is: "$x_o \in \{ x \in X : P(x) \} \Rightarrow P(x_o)$" is a true mathematical statement. So, even if "$x_o \in \{ x \in X : P(x) \}$" is false, the whole implication is true!
Two: Suppose that we are working with some element, x_o, of X, and that we need to know whether $x_o \in \{ x \in X : P(x) \}$, or $x_o \notin \{ x \in X : P(x) \}$. The way to do it is, do the work necessary to find out the truth-value of $P(x_o)$. If $P(x_o)$ is true, then x_o meets the criterion for membership in $\{ x \in X : P(x) \}$. If $P(x_o)$ is false, then x_o meets the criterion for NOT belonging to $\{ x \in X : P(x) \}$. Indeed, if $P(x_o)$ is false, then x_o meets the criterion FOR belonging to the complement, $\{ x \in X : P(x) \}^c$, of $\{ x \in X : P(x) \}$.

(2.9) Exercise: Show that $\{ x \in X : \sim P(x) \} = \{ x \in X : P(x) \}^c$.

SET OPERATIONS: INTERSECTION, UNION, DIFFERENCE, SYMMETRIC DIFFERENCE, COMPLEMENT

There are several operations used to combine sets and thereby form new sets. First of all, we work with sets that are subsets of some given set to perform these operations. So let's let X be the name of the set that all our sets are subsets of. Suppose A and B are arbitrary subsets of X. Here are the definitions of the three basic operations:

The **intersection** of A and B, denoted, in symbols, by $\mathbf{A \cap B}$, is the set of all the elements of X that belong to both A and B. In set selector notation,
$A \cap B = \{ x \in X : x \in A \text{ and } x \in B \}$. Please be sure you agree this means the same as the definition! Also, notice the similarity between the symbols for intersection and the logic operator "and."

Two sets A and B are **disjoint** if $A \cap B$ is the empty set: there are no elements of X that are in both sets at once.

The **union** of A and B, denoted, in symbols, by $\mathbf{A \cup B}$, is the set of all the elements of X that belong to at least one of A and B. In set selector notation,
$A \cup B = \{ x \in X : x \in A \text{ or } x \in B \}$. Please be sure you agree this means the same as the definition! Also, notice the similarity between the symbols for union and the logic operator "or."

The (**set**) **difference** between A and B, denoted, in symbols, by $\mathbf{A \sim B}$, is the set of all the elements of X that belong to A and do not belong to B. In set selector notation,
$A \sim B = \{ x \in X : x \in A \text{ and } x \notin B \}$. Please be sure you agree this means the same as the definition!

The **symmetric difference** of A and B, denoted, in symbols, by $\mathbf{A \Delta B}$, is the set of all the elements of X that belong to exactly one of A and B. In set selector notation,
$A \Delta B = \{ x \in X : (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$. Please be sure you agree this means the same as the definition! Note the similarity with "exclusive or."

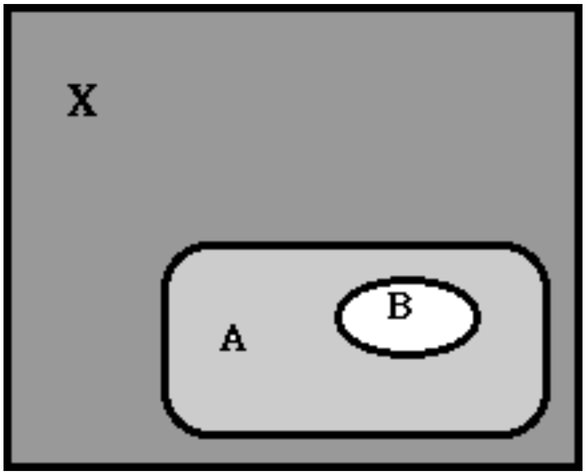
(2.10) Exercise: Use the paradigm in the "important note on set equality" to verify the equation
$$A \Delta B = (A \sim B) \cup (B \sim A).$$

There is also a set operation that can be symbolized as a unary operation. Let A be a subset of X.

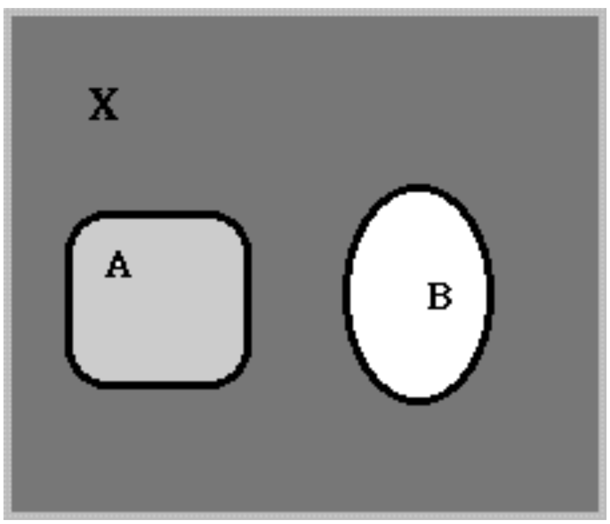
The **complement** of A, denoted $\mathbf{A^c}$, or $\mathbf{\sim A}$, is the set of all x in X that are not elements of A. In set selector notation, $\sim A = \{ x \in X : x \notin A\}$. Please be sure you agree this means the same as the definition! Note the similarity with "not."

(2.11) VENN DIAGRAMS

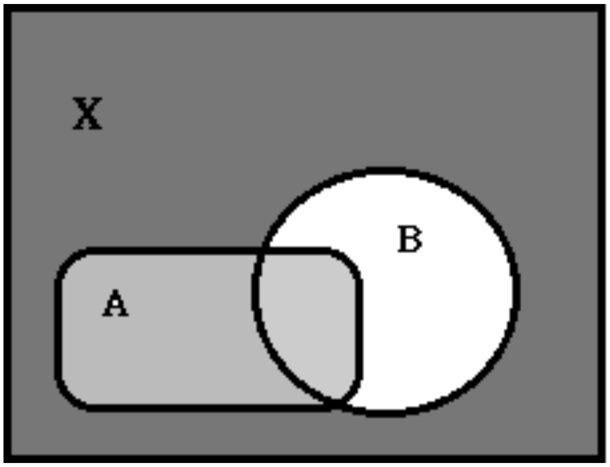
The way sets fit together - or don't - can be illustrated in some cases with **Venn diagrams**. These are pictures that show the various possibilities. Here are some for two sets:

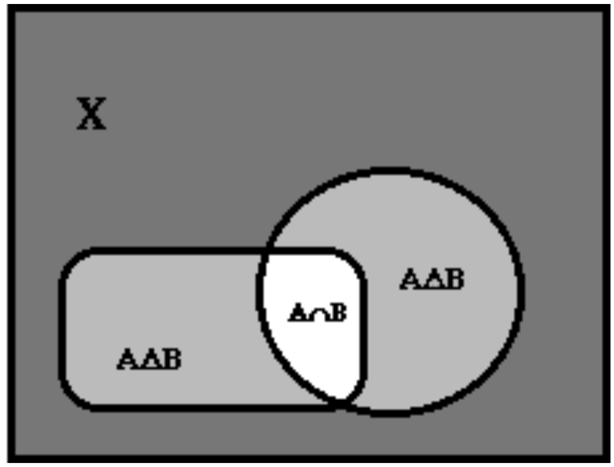


This Venn diagram illustrates "B is contained in A," in symbols, "$B \subseteq A$."
It also illustrates "A contains B," in symbols, "$A \supseteq B$."



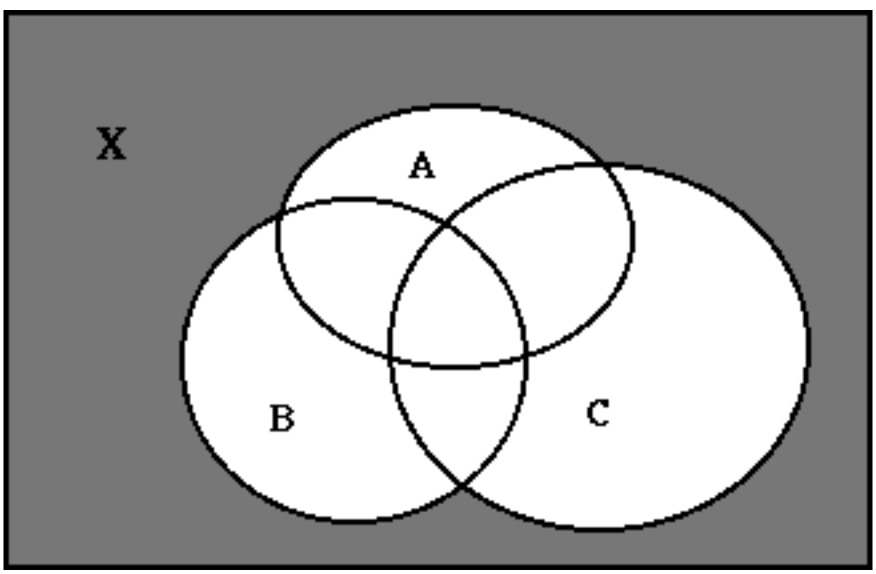
This Venn diagram illustrates "A and B have no elements in common," or "A and B are disjoint," in symbols, "$A \cap B = \varnothing$."





The Venn diagram on the left illustrates the most general situation, for neither A nor B contains the other, and the sets have some elements in common. The Venn diagram on the right shows the intersection and the symmetric difference of A and B. Please note which symbols are which!

The next Venn diagram shows three sets. Each oval represents one set.



(2.12) Exercises: The ovals divide the set X into how many parts? Express each part in terms of set operations. For example, the curvilinear triangle in the middle would be labeled $A \cap B \cap C$. Legalistically, this should be written $A \cap (B \cap C)$ or $(A \cap B) \cap C$, but these two sets, as well as $A \cap (C \cap B)$, are all equal to each other. But $A \cup (B \cap C)$ and $(A \cup B) \cap C$ *are* different! Find these sets, as well as $A \cap (B \cup C)$, in the Venn diagram.

(2.13) Exercise: Use the paradigm in the "important note on set equality" to verify the equation $\sim A = X \sim A$.

Mathematical statements that involve sets are made in a context. There is always a set in the background somewhere that contains all the sets we are working with. Often that "**universal**" set, or "**universe of discourse**," is not specified. This can cause confusion. So please ask what set the objects under discussion belong to!

A fundamental assumption, first version: ***If A and B are sets that are not contained in the same universal set, then a set exists that contains all the elements in A and in B, and no other mathematical objects. This set is called the union of A and B, and is denoted*** $A \cup B$.

The new set, whose existence is guaranteed by this assumption, is a new universal set, as far as A and B are concerned. The notation is the same as though there were a universal set in the background, comprised of all the elements in A and in B. In fact, this assumption is there so that we can *create* convenient universal sets. This is the first version because we also need to be able to form a universal set that contains all the elements in an arbitrary set of sets.

(2.14) Problem: Suppose A and B are subsets of a set X, but do not assume they are disjoint. How might you create new sets A^* and B^* that would be "copies" of A and of B, and disjoint from each other and from X?

(2.15) Exercise: Suppose you have two sets A, B that have nothing to do with each other (like A^* and B^* in the previous exercise). For each a in A, and each b in B, construct the set $(a, b) := \{ \{a\}, \{a, b\} \}$. Verify, perhaps by using the important note on set equality, that
"$(a, b) = (c, d)$" is true if, and only if, "$a = c$ and $b = d$" is true. We will be using these "ordered pairs" in a very basic way. This exercise shows that ordered pairs can be defined as sets.