LOGIC AND SETS

THE DENIAL OF A STATEMENT WITH QUANTIFIERS

One of the main reasons for introducing you to logic is so that you can work with mathematical statements knowledgeably. Sometimes it is easier to understand the denial of a mathematical statement than it is to understand the mathematical statement itself. But how do we transform a complicated mathematical statement into its denial? The way to work out the denial of a statement with quantifiers is easy to say, not always easy to do, especially when we get past all the quantifiers. The rules are simple: replace $\sim(\forall x \in X)$ with $(\exists x \in X)\sim$, and replace $\sim(\exists x \in X)$ with $(\forall x \in X)\sim$. That is, replace the quantifier by the other kind, and move the $\sim$ to the other side.
Notice: do not make any changes in the $x \in X$ statement! After all, this is still going to be a statement about the elements of the set X! Let's apply these rules to an example, then explain how it works.
Example: Express the denial of $(\forall x \in \mathbf{N})(\exists y \in \mathbf{N})(\forall z \in \mathbf{N})(z + y > x)$ in a form that has no negation signs ($\sim$). Don't worry about what the mathematical statement means, or what its truth-value is.

Solution: One step at a time: The denial of the original mathematical statement is $\sim(\forall x \in \mathbf{N})(\exists y \in \mathbf{N})(\forall z \in \mathbf{N})(z + y > x)$. We replace the $\sim(\forall x \in \mathbf{N})$ part by $(\exists x \in X)\sim$, to get $(\exists x \in \mathbf{N})\sim(\exists y \in \mathbf{N})(\forall z \in \mathbf{N})(z + y > x)$. There is still a $\sim$, so we replace the $\sim(\exists y \in \mathbf{N})$ part by $(\forall y \in \mathbf{N})\sim$, and get $(\exists x \in \mathbf{N})(\forall y \in \mathbf{N})\sim(\forall z \in \mathbf{N})(z + y > x)$. There is still a $\sim$, so we replace the $\sim(\forall z \in \mathbf{N})$ part by $(\exists z \in \mathbf{N})\sim$, and get $(\exists x \in \mathbf{N})(\forall y \in \mathbf{N})(\exists z \in \mathbf{N})\sim(z + y > x)$. There is still a negation sign. But we are past the quantifiers, so the rules don't help. We have to figure out how to negate $(z + y > x)$ without using a negation sign. We know how: $\sim(z + y > x)$ is logically equivalent to $(z + y \quad x)$. Thus, the desired statement is $(\exists x \in \mathbf{N})(\forall y \in \mathbf{N})(\exists z \in \mathbf{N})(z + y \quad x)$.

Though it's not part of the problem or the solution, we note that this last statement is false, for $\mathbf{N}$. That means the statement we started with is true, for $\mathbf{N}$.

Now, to explain how this works, $\sim(\forall x \in X)P(x)$ is a mathematical statement, and $P(x)$ is a statement with a free variable — $P(x)$ says something about an x in X without specifying which x, so $P(x)$ has no truth-value. The quantifying phrase $(\forall x \in X)$ says nothing about any x, what $(\forall x \in X)$ says is that the expected phrase about x that follows is true for each and every x in X. Thus $(\forall x \in X)P(x)$ asserts that, for every x in X, $P(x)$ is true. The denial of this, namely $\sim(\forall x \in X)P(x)$, therefore asserts that it is not true that, for every x in X, $P(x)$ is true. Now this means that, as x takes on all the values it can in the set X, each $P(x)$ becomes, for that value of x, a mathematical statement, not a statement with a free variable, and the denial asserts that these mathematical statements are not all true. That means that at least one of them must be false. Otherwise, they would all be true, and the universal quantifier would yield a truth-value of TRUE for the universal AND, namely $(\forall x \in X)P(x)$. Accordingly, if it is true, the denial says there is at least one x in X, call it x_o, such that $P(x_o)$ is false, or, that $\sim P(x_o)$ is true, for x_o. We have come to the conclusion that, if $\sim(\forall x \in X)P(x)$ is true, $(\exists x \in X)\sim P(x)$ is true too. Let us work backwards. Suppose we know that $(\exists x \in X)\sim P(x)$ is true. Must $\sim(\forall x \in X)P(x)$ be true? Well, the first asserts that, for at least one x in X, $\sim P(x)$ is true. That is, $P(x)$ is false for that x. Hence, it is not true that, for every y in X, $P(y)$ is true, because $P(x)$ is false for a particular x. Thus $\sim(\forall x \in X)P(x)$ is true. What if $(\exists x \in X)\sim P(x)$ is false? The mathematical statement asserts that $\sim P(x)$ is true for at least one x in X. So, in order that $(\exists x \in X)\sim P(x)$ be false, $\sim P(x)$ must be false for every x in X. That means that, for every x in X, $P(x)$ is true. This means $(\forall x \in X)P(x)$ is true. Thus, $\sim(\forall x \in X)P(x)$ is false. Finally, we have seen that $\sim(\forall x \in X)P(x)$ and $(\exists x \in X)\sim P(x)$ are logically equivalent, or, in other words, $\sim(\forall x \in X)P(x) \Leftrightarrow (\exists x \in X)\sim P(x)$.

(3.1) Exercise: Carefully explain why $\sim(\exists x \in X)P(x) \Leftrightarrow (\forall x \in X)\sim P(x)$ is a tautology.

An important convention — truth-values for “empty” quantifiers: $(\boxed{\forall} x \in \varnothing)P(x)$ is *true*; $(\boxed{\exists} x \in \varnothing)P(x)$ is *false!*

(3.2) Exercise: Discuss this convention: try to make sense of it; try to show it is nonsense. What do you want the truth-values to be?

(3.3) DEDUCTION - RULES FOR FINDING TRUTH-VALUES

We know one way to find truth-values: use a truth table. This is often unnecessary. Here are some rules for extracting true mathematical statements from given ones whose truth-values are known. Some of them can be verified by looking at the truth tables in the Mathematical Logic chapter. Those involving quantifiers can be verified by imagining “universal **and**’s,” or “universal **or**’s.”

Rule 1a: If “A and B” is true, then “A” is true (from the truth of A and B we deduce the truth of A).
Rule 1b: If “A and B” is true, then “B” is true (from the truth of A and B we deduce the truth of B).

Rule 2: If "A or B" is true, and "A" is false, then "B" is true.

Rule 3a: If "A or B" is false, then "~A" is true.
Rule 3b: If "A or B" is false, then "~B" is true.

Rule 4 (The Law of the Excluded Middle): If "~A" is false, then "A" is true.

Rule 5: If $(\forall x \in X)P(x)$ is true, and x_o is an element of your choice in X, then $P(x_o)$ is true.

Rule 6: If $(\exists x \in X)P(x)$ is true, then we can, in principle, find an x_o in X, not of our choice, about which we know only that $P(x_o)$ is true.

Rules 5 and 6 are the rules most often overlooked or misused in proving theorems. This often happens because quantifiers are often left out! Always look for quantifiers, if there are variables!

Rule 7a: If "$A \Rightarrow B$" is true, and "A" is true, then "B" is true.
Rule 7b: If "$A \Rightarrow B$" is true, and "~B" is true, then "~A" is true.

Rule 8: If "$A \Rightarrow B$" is false, then "A" is true, and "~B" is true.

Rule 9 (Substitution Rule): If P(x) is a mathematical statement with a free variable, and P(a) is true for some a, then, if b = a, P(b) is true too.
$(\forall x \in X)P(x,x)$,

Example: Given that $A \Rightarrow B$ is true, and that B is false, what can we deduce about the truth-value of A?

Solution: Here is the truth table for $A \Rightarrow B$, (1.11):

A	**B**	**A⇒B**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

The lines with $A \Rightarrow B$ true are lines 1, 3 and 4. Of these, only line 4 has B false. Thus, we deduce, using the truth table, that A is false.

Example: Suppose that X is a non-empty set.

Given that $(\forall x \in X)(\forall y \in X)(P(x,y) \Rightarrow P(y,x))$, and

$(\forall x \in X)(\forall y \in X)(\forall z \in X)((P(x,y) \text{ and } P(y,z)) \Rightarrow P(x,z))$

are true, deduce that

$(\forall x \in X)(\forall y \in X)(\forall z \in X)((P(x,z) \text{ and } P(y,z)) \Rightarrow P(x,y))$

is true.

Before we start writing down a solution to a problem, we need to have an idea of what to do. The mathematical statement we are to prove true resembles the second of the given ones. And if we were to recopy the first one, but change the variable names, "x" to "y," and "y" to "z," we could replace the "P(y,z)" in the third statement to "P(z,y)" and then use the second given statement, with some re-naming, to complete the work. So the Solution that comes next will consist of a step-by-step account of how to do that, using the Rules for deduction.

Solution: This will be much longer than necessary! It contains lots of explanation that you don't have to put into your answers to the Exercises that follow.

Suppose that x_o, y_o and z_o are arbitrary elements of X. This means that we have to choose the three elements "at random," or allow anyone at all to choose them.

We have to prove that, for these particular values of x, y and z, "$(P(x,z) \text{ and } P(y,z)) \Rightarrow P(y,z)$" is true. Notice that, because they have been *chosen*, x_o, y_o and z_o are not variables.

The statement "$(P(x_o,z_o) \text{ and } P(y_o,z_o)) \Rightarrow P(x_o,y_o)$" has the form $A \Rightarrow B$.

Therefore, if $A :=$ "$P(x_o,z_o) \text{ and } P(y_o,z_o)$" is false, "$(P(x_o,z_o) \text{ and } P(y_o,z_o)) \Rightarrow P(x_o,y_o)$" is vacuously true, according to the truth table for $A \Rightarrow B$.

On the other hand, if "$P(x_o,z_o) \text{ and } P(y_o,z_o)$" is true, we have to show that "$P(x_o,y_o)$" is true, in order to show that "$(P(x_o,z_o) \text{ and } P(y_o,z_o)) \Rightarrow P(x_o,y_o)$" is true, by the truth table for $A \Rightarrow B$.

Thus, assume that "$P(x_o,z_o) \text{ and } P(y_o,z_o)$" is true.

By Rule 1a, we deduce that "$P(x_o,z_o)$" is true.

By Rule 1b, we deduce that "$P(y_o,z_o)$" is true.

We are given that $(\forall x \in X)(\forall y \in X)(P(x,y) \Rightarrow P(y,x))$ is true.

We choose $x := y_o$, and deduce, by Rule 5, that $(\forall y \in X)(P(y_o,y) \Rightarrow P(y,y_o))$ is true.

We choose $y := z_o$, and deduce, by Rule 5, that "$P(y_o,z_o) \Rightarrow P(z_o,y_o)$" is true.

Now we have deduced that "$P(y_o,z_o)$" and "$P(y_o,z_o) \Rightarrow P(z_o,y_o)$" are true.

By Rule 7a, we deduce that "$P(z_o,y_o)$" is true.

We have deduced that "$P(x_o,z_o)$" is true, and that "$P(z_o,y_o)$" is true.

By the truth table for "AND," we deduce that "$P(x_o,z_o)$ and $P(z_o,y_o)$" is true.

We are given that $(\forall x \in X)(\forall y \in X)(\forall z \in X)((P(x,y) \text{ and } P(y,z)) \Rightarrow P(x,z))$ is true.
By using Rule 5 three times, we can choose $x := x_o$, $y := z_o$, and $z := y_o$ and deduce that "$(P(x_o,z_o) \text{ and } P(z_o,y_o)) \Rightarrow P(x_o,y_o)$" is true.
By Rule 7a, we deduce that "$P(x_o,y_o)$" is true.
We have shown that our statement, that has the form $A \Rightarrow B$ for each choice of variables, is true.
Since the variables can now take on any values, and yield a true statement, we are done.

Example: Suppose that X is a non-empty set.
Given that $(\forall x \in X)(P(x) \Rightarrow Q(x))$, and

$(\exists x \in X)(R(x) \Rightarrow P(x))$

$(\forall x \in X)R(x)$

are true, deduce that

$(\exists x \in X)Q(x)$

is true.

Let us use a "contradiction argument." This means we *assume* that the mathematical statement we want to show is true is in fact *false*. Then we work with the new statement, $(\forall x \in X)(\sim Q(x))$, which is the denial of the original one, and all the others, and try to deduce that a contradiction is true, namely a statement that always has the truth-value FALSE. Then we apply Rule 4 to show that our original assumption, that $(\forall x \in X)(\sim Q(x))$ is true, is not correct, hence that the original statement is true, after all.

For now, we assume that $(\forall x \in X)(\sim Q(x))$ is true. By Rule 6, we can find x_o in X such that $R(x_o) \Rightarrow P(x_o)$ is true. By Rule 5, applied to the statement $(\forall x \in X)R(x)$, with x_o as the element of our choice, $R(x_o)$ is true. Therefore, $P(x_o)$ is true, by Rule 7a. By Rule 5, applied to the statement $(\forall x \in X)(P(x) \Rightarrow Q(x))$, with x_o as the element of our choice, $P(x_o) \Rightarrow Q(x_o)$ is true. Thus the contrapositive, $\sim Q(x_o) \Rightarrow \sim P(x_o)$ is true. By Rule 5, applied to the statement $(\forall x \in X)(\sim Q(x))$, with x_o as the element of our choice, $\sim Q(x_o)$ is true. Therefore, $\sim P(x_o)$ is true, by Rule 7a. We have shown that "$P(x_o)$ and $\sim P(x_o)$" is true, and this is a contradiction, that is, has the truth-value FALSE, because, by the truth table for NOT, $P(x_o)$ true means $\sim P(x_o)$ is false. Thus, our *assumption*, that $(\forall x \in X)(\sim Q(x))$ is true is false. Therefore, its denial is true, namely $(\exists x \in X)Q(x)$.

This was only an example of a contradiction argument. Sometimes we *need* to make a contradiction argument, because assuming the contrary gives us a statement to work with as true. Usually, the denial will be an *existence* statement, so we'll have an object to work with. Here, the original statement was an existence statement, so its denial was a "universal AND" statement. Indeed, we can give a slightly shorter proof without the contradiction! Here it is:

By Rule 6, we can find x_o in X such that $R(x_o) \Rightarrow P(x_o)$ is true. By Rule 5, applied to the statement $(\forall x \in X)R(x)$, with x_o as the element of our choice, $R(x_o)$ is true. Therefore, $P(x_o)$ is true, by Rule 7a. By Rule 5, applied to the statement $(\forall x \in X)(P(x) \Rightarrow Q(x))$, with x_o as the element of our choice, $P(x_o) \Rightarrow Q(x_o)$ is true. By Rule 7a, recalling that $P(x_o)$ is true, we deduce that $Q(x_o)$ is true. Therefore, by definition of the existential quantifier, $(\exists x \in X)Q(x)$ is true.

In the following Exercises, justify your answers by citing appropriate rules or truth tables, or by constructing a suitable example.

(3.4) Exercise: Show that, from the truth of A and B, we can deduce the truth of A or B, but not vice versa.

(3.5) Exercise: Formulate versions of Rules 5 and 6 that apply to statements of the form $(\forall x \in X)(\forall y \in X)P(x,y)$ and $(\exists x \in X)(\exists y \in X)P(x,y)$, and explain why they work.

(3.6) Exercise: Given that $A \Rightarrow B$ is true, and that A is false, what can we deduce about the truth-value of B?

(3.7) Exercise: Given that $A \Rightarrow B$ is true, and that A is true, what can we deduce about the truth-value of B?

(3.8) Exercise: Given that $A \Leftrightarrow B$ is true, and that A is true, what can we deduce about the truth-value of B?

(3.9) Exercise: Given that $A \Rightarrow B$ is true, and that B is true, what can we deduce about the truth-value of A?

(3.10) Exercise: Given that $P(x_o)$ is true for some x_o in X, what can we deduce about the truth-value of $(\exists x \in X)P(x)$?

(3.11) Exercise: Given that $P(x_o)$ is true for some x_o in X, what can we deduce about the truth-value of $(\forall x \in X)P(x)$?

(3.12) Exercise: Given that $P(x_o)$ is false for some x_o in X, what can we deduce about the truth-value of $(\forall x \in X)P(x)$?

(3.13) Exercise: Given that $(\forall x \in X)P(x)$ is false, what can we deduce about the truth-value of $P(x_o)$ for some x_o in X?

(3.14) Exercise: Given that $(\forall x \in X)P(x)$ is false, what can we deduce about the truth-value of $P(x_o)$ for every x_o in X?

(3.15) Exercise: Given that $(\forall x \in X)(P(x) \Rightarrow Q(x))$ is true, and that $(\exists x \in X)P(x)$ is true, what can we deduce about the truth-value of $(\exists x \in X)Q(x)$?

A fundamental assumption: ***A statement that is formed by compounding mathematical statements, each of which is a well-defined mathematical statement, has a truth value, TRUE or FALSE.***