INTRODUCTION

When you read a novel, attend a play, or go to a movie, you are asked, at least implicitly, to "suspend disbelief." For the moment, you accept the world as it is in the author's mind. In this course, I ask you to "suspend belief" in all the mathematics you've seen to date! I don't mean for you to forget it, though! And I especially want you to keep all your intuition and skill, especially in geometric visualization, algebraic manipulation and recognition of expressions that are perfect squares, differences of squares, reciprocals of reciprocals, quadratics that factor, and so on. What I want you to do is to suspend belief in the "proofs" you have seen, in the explanations, and derivations, in the "whys" about mathematical things you've seen before. It's not that they are wrong, it's that they may be disjointed, they may draw on several kinds of mathematics, they may rely on concepts from everyday experience, but they were, most likely, not based on deductions from mathematical statements ***known*** to be true. The arguments you have seen were intended to convince you, to give you a sense of understanding "what was going on." They were not proofs, though, based on what you knew at the time. One example of this sort of not-quite-a-proof is probably in your calculus book where you learned that $\frac{\sin x}{x}$ has limit 1 as x approaches 0. These arguments are true, all right, but they often draw on concepts and properties of area or arc length that you had not previously seen proved. The arguments are often vague about the connection between angle and number and area.

We are going to rebuild Calculus by basing it on a some basic rules and assumptions, and by proving everything we do. We are going to *prove*, among many other things, that $\frac{\sin x}{x}$ does have limit 1 as x approaches 0.

TERMS

We will use **terms** in **mathematical statements**. We will also use some terms, such as "variable," to describe symbols that occur within mathematical statements. There are two kinds of terms: undefined, and defined. **Undefined terms** are words or phrases that you are assumed to know a meaning for, such as "set," "one," "variable," and so on. We can *describe* the meanings of such terms with examples. In some cases, we simply *introduce* an undefined term. Mathematical terms will be put in bold type when first *officially* introduced. No mathematical terms are officially introduced in this Introduction, despite their appearance in bold type. **Defined terms** are words or phrases that are introduced by a formal **definition**. We try to make the number of undefined terms small. One very important thing for you to do is to learn every definition! Each definition gives us one of the mathematical objects that we will study. The better you know a

definition, the less time you have to spend looking it up. Most terms are defined in terms of other defined terms. So if you know them all, you'll have less trouble understanding new ones.

MATHEMATICAL OBJECTS

We will be dealing with "mathematical objects." This is one of our "undefined terms," something you "already know." It is also a very vague term. We use words, symbols and pictures to communicate about mathematical objects. Mathematical objects are constructions of the mind. You won't find any in the world of sensory experience. You will find plenty of inspiration for mathematical objects there, and in your own experiences of the everyday world. Usually our mathematical objects will be numbers, or sets of numbers, or related constructions, all describable in terms of sets. But they don't have to be. Anything that you can think of, that has a definite meaning for you, is potentially a mathematical object. Please notice that sets are mathematical objects. The term "set" is also an undefined term. You won't find sets in nature either. For us, the mathematical objects of primary interest are those that arise in Calculus. Calculus is built in terms of several systems of numbers, and functions, and mathematical processes, such as limit and integral. All these are mathematical objects.

MATHEMATICAL STATEMENTS

In this course you will learn to work with **mathematical statements**. Roughly speaking, these are statements that have a definite truth-value, TRUE or FALSE. Each mathematical statement is an assertion, usually involving equality, inequality, membership in a set, or inclusion of one set in another. A mathematical statement may assert that some (already defined) mathematical object possesses some (already defined) property. That is, mathematical statements are about mathematical objects, and they always assert something. In some branches of mathematics, mathematical statements are mathematical objects. For us they won't be.

Sometimes a statement refers to one or more unspecified members of a **set**. An example is "$x = y$." These unspecified elements, x and y in the example, are called **variables**. The statement "$x = y$" is not *quite* a mathematical statement, because it does not have a definite truth-value. However, when actual values are substituted for the variables x and y, "$x = y$" does have a definite **truth-value**, **TRUE** or **FALSE**. In the example, "$x = y$," let's say that x and y belong to the set of positive integers. Then, when we choose $x = 1$, $y = 2$, the statement becomes "$1 = 2$," which "is false," meaning "has truth-value FALSE." When we choose $x = 3$, $y = 3$, the statement becomes "$3 = 3$," which is true, because it has the truth-value TRUE.

More will be said about sets and variables later.

Every term in a mathematical statement needs to have a definite meaning. The statement, "It is rainy here," is a "non-example." What does "rainy" mean? Where is "here?"

Another aspect of having a definite meaning is that variables or objects referred to in mathematical statements will each have to belong to a definite set, called a "**Universe of Discourse**." There may be several Universes of Discourse if several sets are involved in a statement. We won't always follow this rule literally. The missing references to the sets that contain the variables we work with are supposed to be clear "by context."

Mathematical statements will be expressed in several formats, ranging from ordinary English to virtually unreadable strings of special "**logic**" symbols. If you work patiently and persistently you can probably learn the skills for manipulating mathematical statements. You can learn how to translate them from one format to another. You can learn to express them informally. You can supply missing pieces, or take pieces out to make a mathematical statement easier to remember. The most common kind of missing piece in a mathematical statement is a "**quantifier**," usually rendered by "**for all**" or "**there exists**." Quantifiers tell us how to interpret statements that have variables in them. Quantifiers will be discussed in detail later.

We will also talk *about* mathematical statements. Many statements made, about mathematical statements, in this course won't be mathematical statements themselves. Comments, explanations, suggestions, instructions, are examples of such non-mathematical statements. Extremely valuable stuff: ideas, intuitions, guesses, strategies, creative mistakes, and the like, are not mathematical statements either, but they are to a great extent the "real stuff" of mathematics. Will you come to experience these things? I hope so!

We will start with beginning **mathematical logic**, with **quantifiers**, and basic **set theory** - not much - just what we need to rebuild Calculus. The rebuilding of Calculus is the main theme of the course. Another theme is problem-solving. In addition to the algebraic skills you already have, you need to develop skills in using inequalities to simplify formulas, skills in using logical operations, skills in "recognition." Some of your algebraic skills may be "rusty." If so, you will suffer needlessly if you do not practice with the help of algebra problems, even tutors! No skills in using logical operations are assumed. Some skill in recognition, such as knowing that, for numbers, $a^2 - b^2$ is equal to $(a - b)(a + b)$, is assumed.

We are going to take a journey through an Advanced version of first- and second-year Calculus, via the "Rigorous Approach." Along the way we'll come across many things you probably did not see in your first trip. This will provide a little relief for the hardships of a strange journey to the top of a mountain that we build as we climb!

We will assemble our logical tools in the abstract realm of Sets and **Functions**, and place in it a particular setting, the **Set of Natural Numbers**. The Natural Number System will be our foundation. We will assume that certain mathematical statements, the **Peano Postulates**, are true. Using these mathematical statements, we will define the operations of addition and multiplication, define the relation "less than or equal to" and deduce some properties of these defined mathematical objects. Our tools will allow us to construct other sets using the Natural Numbers and copies of it. We will then construct the **integers**. Then, in "quick" succession, we'll construct the **rational numbers**, the **real numbers**, and the **complex numbers**.

As we construct the systems of numbers mentioned before, we will deduce a large number of properties of them that are important for the Advanced approach to Calculus. In particular, the Trichotomy Law, the Well-Ordering Theorem for the Natural numbers, the Completeness Property for the Real Number System, and topological concepts will be very important. Many other objects will be constructed along the way, and their properties developed, that are needed to understand Calculus from an Advanced point of view.

Each of the mathematical objects we construct will be a list of **axioms**, **definitions**, and **theorems** based on previous axioms, definitions and theorems. These will all be statements or mathematical statements. Axioms are mathematical statements that we *assume* are true. Definitions are statements (or sets of them) with free variables, that become true or false mathematical statements when values are assigned to each of the free variables in the definition. If all the statements of a definition are true when certain values are "plugged into" their free variables, we say the definition is "satisfied" by those values of the variables. A definition usually assigns meaning, in terms of the statements in it, to a term that has not previously been defined. Theorems are statements that may have free variables, that are true statements no matter what ("legal") values are assigned to the free variables.

The process of finding the truth value of a mathematical statement involves making a step-by-step list of mathematical statements, organized by other statements, not necessarily mathematical, that specify the reasons for the truth or falsity of particular mathematical statements. This is usually called a process of **deduction**, or an **argument**. The argument may, or may not, be correct. If it

is correct, it is called a **proof**. It "is" correct if knowledgeable persons agree that the truth value of the mathematical statement in question has been found.

One of the prime goals of this course is for you to become one of these knowledgeable persons. Then you won't have to wait for a grader or your teacher to tell you whether your argument is a proof - *you will know!* This happens over a period of time, but suddenly, you catch on; perhaps it's like learning to ride a bicycle...

Our standard for *"mathematical truth"* is not that we believe a statement to be true - that we'll call a *conjecture* - but that we believe that it is proven. We can also *accept* a statement as true; we call an accepted-as-true statement an **axiom**, or **postulate**.

We will constantly be making **definitions**. These are statements that introduce a term, and equate it to some statement, or even a set of statements, all involving terms already known. **It is very important for you to make a list of definitions, and commit each one to memory**!

There will be many **exercises** and **problems** - exercises are intended to be easy to do, to help you become familiar with the concepts, notation, terms, and so on. Problems ask you to bring different things together. You will be asked to make proofs. There are problems that demand skill in dealing with logic. Problems also call on all your other skills, because the statements being proved have to do with numbers, sets of numbers, functions, formulas, and so on, and you need to manage these along with the logic.

When we are given a true mathematical statement, we can make choices that exploit its truth. When we are given a mathematical statement to prove, we imagine that others made the choices, and we must verify that the mathematical statements resulting from their choices are true.

HOW THIS TEXT IS DESIGNED TO WORK

☞ this section can be skipped; it is a guide to the format of this text

Many students have told me, in effect, that you need a chatty text, one with lots of examples, carefully worked out. There are some drawbacks — it means a lot to read!

So I have attempted to vary the way the text is laid out, so you can skip some stuff. **Comments** look like this. They are not part of the main line of presentation. Some people might find them helpful, though.

In many places, especially in important, difficult arguments, a lot of explanation will be put in.

This can obscure the mathematical content. So some of the explanatory material will be written in smaller type, and sometimes (when it involves clarification) it will be put in parentheses. On the other hand (especially when an argument has parallel parts), **a signpost phrase** will appear in bold type. These allow a quick overview of the way an argument is organized.

The material can be broken into chapters and sections, but I will divide it even more, using bold headings in all capital letters, e.g., **HOW THIS TEXT MATERIAL IS DESIGNED TO WORK**, that identify the main topic of a few paragraphs. Sometimes,

☞ **the headings are bold, in small letters, to identify a "sub-topic"**

The text is so chatty you might lose track otherwise — there will be many tangents!

Some stuff, written like this, you should not only NOT skip, you should think about it, and be warned to look out for it to crop up!

Exercises: Will look like this.

Examples: Will look like this.

Definitions will be introduced in two ways. At first, especially, most definitions will be definitions of terms that are basic, or will be natural names for concepts you are assumed to know a meaning of. Such **terms** will appear in bold type. You should make yourself a list of them. I could put them in a list for you, but most people benefit by the activity of writing them down! The other way for definitions to be introduced, especially if they are made in terms of other defined terms, is illustrated by an

Example:

(0.1) ***Definition:*** An **axiom**, or **postulate**, is a mathematical statement that is assigned the truth-value TRUE.

Compare this format to the one used to define these synonymous terms in an earlier paragraph.

Symbols are used a lot, to make things brief, and more understandable. We define the meaning of a symbol by setting **notation**. Most of the time, notation will appear as part of a definition, and then it will look like this:

(**Example**: We could have put part of the previous discussion this way)

(0.2) ***Definition and notation:*** The **truth-value** of a mathematical statement is denoted by **TRUE** or by **FALSE**. If the mathematical statement is true, its truth-value is TRUE, and if the mathematical statement is false, its truth-value is FALSE.

If a definition, theorem, or example is important enough to have its own heading, it will usually be numbered, sequentially. Exercises and Problems will always be numbered:

(0.3) Example: This is how the numbers look. If this Introduction had more than nine numbered items, but less than 100, the number would have been written (0.03).