

We define $V(y) := x_o + Ty$, where T is a non-singular linear transformation on \mathbb{R}^n , and $x_o \in \mathbb{R}^n$. We then have $V^{-1}(x) = T^{-1}(x) - T^{-1}(x_o)$. For brevity let us write $y_o := T^{-1}(x_o)$, so that $V^{-1}(x) = T^{-1}(x) - y_o$.

We will show that for all measurable f defined on a measurable set E , if $\int_E f$ exists then

$$\int_E f = \int_{V^{-1}(E)} f(V(y)) |\det T| dy.$$

Without loss of generality we may assume $f \geq 0$. If we begin with a measurable set $S \subseteq E$ and let $f(x) := [x \in S]$ (so that f is the characteristic function, or indicator function of S), then

$$\int_E f = |S \times [0, 1]_{n+1}| = |S|.$$

Also, $f(V(y)) = [V(y) \in S] = [y \in V^{-1}(S)]$, so

$$\int_{V^{-1}(E)} f(V(y)) |\det T| dy = \int_{V^{-1}(E)} [y \in V^{-1}(S)] |\det T| dy = |V^{-1}(S)| |\det T| = |T^{-1}(S) - y_o| |\det T|.$$

You have shown that $|T^{-1}(S) - y_o| = |T^{-1}(S)|$ and we have shown that $|T^{-1}(S)| = |\det T^{-1}| |S|$. Thus

$$\int_{V^{-1}(E)} f(V(y)) |\det T| dy = |T^{-1}(S) - y_o| |\det T| = |\det T^{-1}| |S| |\det T| = |S|.$$

Our two integrals are thus equal in this case.

Now the formula (5.4) for the integral of a simple function, and Theorem (4.13), that gives a sequence of simple measurable functions that increases to a given non-negative function pointwise, together with Monotone convergence, show that the desired formula holds.

Remark: “Literal Substitution” It is useful in sums and integrals to make a change of variables that preserves the dummy variable, at least in name. For example, in checking the summation-by-parts formula or the equation

$b^n - a^n = (b-a) \sum_{j=0}^{n-1} b^{n-1-j} a^j$, two sums with the same dummy variable are needed. We can treat sums as integrals.

The formula

$$\int_E f = \int_{V^{-1}(E)} f(V(y)) |\det T| dy.$$

can be derived (remembered!) by “literal substitution” as follows:

$$\begin{aligned} \int_E f &= \int_{x \in E} f(x) dx \\ &= \int_{V(x) \in E} f(V(x)) dV(x) \\ &= \int_{x \in V^{-1}(E)} f(V(x)) |\det V'(x)| dx \\ &= \int_{V^{-1}(E)} f(V(x)) |\det T| dx \end{aligned}$$

in our case since $V'(x) = T$. When we work with sums, our change of variables must be integer-valued. Caution: V has to be one-to-one!