First alternate proof

The idea for our first alternate proof is the Cauchy Inequality $|ab| \le \dfrac{a^2+b^2}{2}$, where a and b are real numbers.

We suppose that v and w belong to an inner product space V. We have to show that $|\langle v, w\rangle| \le \|v\|\|w\|$. We will deal with the case of equality as well. As we did in class, if one of v and w is zero, the result is easy to prove. Hence we assume that neither of v and w is zero. We can find an "angle" θ so that $\langle v, w\rangle = |\langle v, w\rangle| e^{i\theta}$ (if $\langle v, w\rangle = 0$ we choose $\theta = 0$). Then (trick coming!)

$$\begin{aligned}
0 \le \|v - e^{i\theta}w\|^2 &= \|v\|^2 - 2\,\mathbf{Re}\ \left\langle v, e^{i\theta}w\right\rangle + \|w\|^2 \\
&= \|v\|^2 - 2\,\mathbf{Re}\ e^{-i\theta}\langle v, w\rangle + \|w\|^2 \\
&= \|v\|^2 - 2\,\mathbf{Re}\ e^{-i\theta}(|\langle v, w\rangle| e^{i\theta}) + \|w\|^2 \\
&= \|v\|^2 - 2\,\mathbf{Re}\ |\langle v, w\rangle| + \|w\|^2 \\
&= \|v\|^2 - 2|\langle v, w\rangle| + \|w\|^2, \quad \text{so that } |\langle v, w\rangle| \le \frac{\|v\|^2+\|w\|^2}{2}.
\end{aligned} \tag{1}$$

This is not quite what we wanted. So (trick coming!) we *normalize* v and w by defining $v' := \dfrac{v}{\|v\|}$ and $w' := \dfrac{v}{\|w\|}$

(2) and notice that $|\langle v', w'\rangle| \le \dfrac{\|v'\|^2+\|w'\|^2}{2} = 1$ (why?!). But then

$$1 \ge |\langle v', w'\rangle| = \left|\left\langle \frac{v}{\|v\|}, \frac{w}{\|w\|}\right\rangle\right| = \frac{|\langle v, w\rangle|}{\|v\|\|w\|}, \quad \text{and thus } |\langle v, w\rangle| \le \|v\|\|w\|.$$

As done in class, direct calculation shows that *if* $v = zw$ then $|\langle v, w\rangle| = \|v\|\|w\|$. Next, if we are instead only given that $|\langle v, w\rangle| = \|v\|\|w\|$, we notice that this means that $|\langle v', w'\rangle| = \|v'\|\|w'\| = 1$. Hence in (2)

$$1 = \frac{\|v'\|^2+\|w'\|^2}{2} = |\langle v', w'\rangle| = e^{-i\theta}\langle v', w'\rangle \quad \text{and therefore}$$

$$0 = \|v'\|^2 + \|w'\|^2 - 2|\langle v', w'\rangle| = \|v'\|^2 + \|w'\|^2 - 2\,\mathbf{Re}\ e^{-i\theta}\langle v', w'\rangle = \|v' - e^{i\theta}w'\|^2. \tag{3}$$

This implies (by "unravelling" (3)) that $v = \left(\dfrac{e^{i\theta}\|v\|}{\|w\|}\right) w$.

Second alternate proof

We *could* have made all this look very clever by dealing with the zero cases first, then letting $A := \|v\|^2$, $B := \|w\|^2$, $C := \langle v, w\rangle$ and then expanding $0 \le \|Bv - Cw\|^2$. The "cross term" partly cancels, leaving $0 \le B(AB - |C|^2)$. This is how it's done in *Principles of Mathematical Analysis,* Third Edition, by Walter Rudin.