Introduction

"Binary Search" really refers to a *method* for finding a number with a desired property, or else demonstrating that such a number does not exist. For example, how does we find a needle in a haystack? Maybe we'd have a strong magnet, knowing that the needle is made of steel, and we'd need permission to dismantle the haystack. But we'd have to reassemble it, I suppose. First, we'd divide the stack in half, then use the magnet to find out which half the needle is in. If we could not tell, we'd divide both halves in half. If we could, we'd divide the half with the needle in half again, use the magnet, and keep on until we find it. That's the whole idea.

We'll use the Binary Search Method in some examples.

Finding the supremum and infimum of a non-empty bounded set

Let S denote our nonempty bounded set. We are given that there exist numbers A and B that bound the set S. That is, for every x in S, $B \leq x \leq A$. Let's assume that $B < A$. Otherwise, S has only one point, and we don't have to look for our sup and inf! We know by the Completeness Axiom that the number $\sup S$ exists. We need to find it.

Since S is nonempty we know that at least one of the intervals $[B, (B+A)/2]$ and $[(B+A)/2, A]$ contains a point of S (why?). We're looking for big elements of S so we choose the right-most of the two intervals. To avoid complicated long expressions like $(B+(B+A)/2)/2$ we'll use a notational convention in our search: we'll let our original interval be denoted $[L_1, R_1]$. Then, if $[(B+A)/2, A] \cap S \neq \emptyset$, we'll write $[L_2, R_2]$ for $[(B+A)/2, A]$. Otherwise, $[L_2, R_2]$ will stand for $[B, (B+A)/2]$. It's important to notice that $L_1 \leq L_2$ and that $R_2 \leq R_1$. It's not as important to notice that exactly one of the statements "$L_1 = L_2$" and "$R_1 = R_2$" is true. *It is very important* to notice that $R_2 - L_2 = (R_1 - L_1)/2 = (A-B)/2$.

We now repeat this process, dividing our chosen interval in half. Since the chosen interval contained at least one point of S one of the halves of the chosen interval contains a point of S (why?). **We will always choose as our next interval the half that is rightmost.** In this way, we get a sequence of intervals $[L_n, R_n]$. According to our method of choosing intervals, there are no points of S larger than R_n (why?). Thus each R_n is an upper bound of S. The sequence $\{R_n\}$ is a decreasing sequence bounded below by B. The sequence $\{L_n\}$ is an increasing sequence bounded above by A. Thus both sequences $\{L_n\}$ and $\{R_n\}$ converge. And we know (by induction) that $0 < R_n - L_n = 2(A-B)/2^n \to 0$. Thus both sequences have the same limit, say P.

Claim: $P = \sup S$. We need to show that P satisfies the **defining** conditions for being the sup of S. *First:* P is an upper bound for S because if $y > P$ then for all n sufficiently large, $0 \leq R_n < y$, and R_n is an upper bound for S. Thus no element of S can be larger than P (why?), so P is an upper bound for S. *Second:* if $x < P$, then there must exist $L_n > x$ because $L_n \to P$ (the argument is the same one we used when y was bigger than P, but we used the numbers $R - n$ there). But then, since there must be a point of S in $[L_n, R_n]$, there exists a point of S that exceeds x. These two conditions are the conditions that need to be true for a number to be the sup of a set.

Exercise: Decide what has to be changed in the proof so far in order to find the infimum of S, then make the changes and check that you have a proof.

Showing that every real number is the limit of a sequence of rational numbers

We suppose that x_0 is a given rational number. We want to find a sequence of rational numbers that converges to x_0. We may as well assume that x_0 is an irrational number. Otherwise, we could let our sequence $\{x_0, x_0, x_0, \ldots, x_0, \ldots\}$. We will actually construct a sequence of *dyadic* rational numbers r_n that converge to x_0 (the dyadic rational n umbers are those whose denominators are powers of 2. We may also assume that $x_0 > 0$ (why?). We know there exists a natural number n_0 larger than x_0 (why?). This means that $x_0 \in I_0 := [0, n_0]$. We divide I_0 in half. One of the halves contains x_0. Both halves cannot, since then x_0 would be the average of two rational numbers, which is a rational number, and we assumed that x_0 is irrational. We choose as I_1 the half of I_0 that contains x_0. Both endpoints of I_1 are dyadic rationals, since each endpoint is either one of the original endpoints or is $n_0/2$.

We now repeat this process, dividing our chosen intervals in half. Since the chosen intervals contain x_0, one of the halves of each chosen interval contains x_0 (why?). **We will always choose as our next interval the**

half that contains x_0. In this way, we get a sequence of intervals $[L_n, R_n]$. As before, the sequence $\{R_n\}$ is a decreasing sequence bounded below by 0 and $\{L_n\}$ is an increasing sequence bounded above by n_0. Moreover, $R_n - L_n = 2n_0/2^n \to 0$, so (again) both sequences converge, and they converge to the same limit. By the Squeeze Principle, since $L_n < x_0 < R_n$ we have $\lim_{n\to\infty} L_n = x_0 = \lim_{n\to\infty} R_n$. We prove by induction that each R_n and each L_n is a dyadic rational. This completes the proof. It's worth noticing that each of our sequences was monotone! This allows us to show that each real number is the limit of a sequence of *irrational* numbers: we just have to show that between any two rational numbers there is a rational multiple of (for example) $\sqrt{2}$.

The Bolzano-Weierstrass Theorem: *Every bounded infinite set of real numbers has a limit point.*

Proof: We have to show this: for every set $S \subseteq \mathbb{R}$ that is a bounded infinite set, there exists a point $y \in \mathbb{R}$ that is a limit point "for" S.

Since S is bounded, there exist numbers A and B such that for every x in S, $B \leq x \leq A$. We define $I_0 := [L_0, R_0] := [A, B]$. We will divide I_0 into its two halves (they overlap at the midpoint), and select as the interval I_1 the left-most half that has infinitely many element of S in it. One of the halves *must* contain infinitely many element of S (why?). We name the left and right endpoints of I_1 L_1 and R_1, respectively.

We then continue the process, always selecting as I_{n+1} the left-most half of I_n that contain infinitely many element of S.

Now $\{L_n\}$ is an increasing sequence bounded above by A and $\{R_n\}$ is a decreasing sequence bounded below by B. Moreover, $R_n - L_n = (A-B)/2^n$, so both sequences converge and they converge to the same limit, say y. Our task now is to verify that y is a limit point "for" S.

Let $\delta > 0$ be given. We need to show that $(y-\delta, y+\delta)$ contains a point of S that is different than y.

Since $\{L_n\}$ is an increasing sequence that converges to y, we know that $L_n \leq y$ for all n. Similarly, $y \leq R_n$ for all n.

We now let δ play the rôle of ϵ in the definition of convergence of $\{L_n\}$ to y. We know there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |L_n - y| = y - L_n < \delta$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2 \Rightarrow |R_n - y| = R_n - y < \delta$. We now select $N := \max\{N_1, N_2\}$.

By our "construction" of the intervals I_n, we know that $I_N = [L_N, R_N]$ contains infinitely many points of S. We will show that $[L_N, R_N] \subseteq (y-\delta, y+\delta)$. Once we do so, we will know that $(y-\delta, y+\delta)$ contains infinitely many points of S, hence must contain two points of S that are different from each other. At least one of them must be unequal to y. This will complete the proof.

To show that $[L_N, R_N] \subseteq (y-\delta, y+\delta)$ we will show that $L_N > y - \delta$ and show that $R_N < y + \delta$.

Now, $N \geq N_1$, so $|L_N - y| = y - L_N < \delta$, so $L_N > y - \delta$.

Exercise: Complete the proof by showing that $R_N < y + \delta$.