The metric space here is ℓ^2, the set of all sequences $x = \{x_n\}$ of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < \infty$.

The metric is $d(x, y) := \sqrt{\sum_{n=1}^{\infty}(x_n - y_n)^2}$.

Exercise: Prove that this is a metric.

The example is the closed unit sphere, namely $S := \{x \in \ell^2 : d(x, 0) = 1\}$.
We also use the notation $\|x - y\| := d(x, y)$.

Exercise: Prove that $d(x, y) = d(x - y, 0)$ **and that** $\|cx\| = |c|\|x\|$ **for all scalars** c **and all** $x \in \ell^2$.

Theorem (Bolzano-Weierstrass): *If T is an infinite subset of a compact set K, then there exists $x_o \in K$ that is a limit point of T.*

Proof: Suppose not. Then no point of K is a limit point of T. Hence every point $x \in K$ has an open neighborhood U_x that contains no point of T except, possibly, x itself. Since K is compact there exist $x_1, \ldots, x_N$ such that $\{U_{x_1}, \ldots, U_{x_N}\}$ covers K Hence $T \subseteq \{x_1, \ldots, x_N\}$, a contradiction since T is infinite.

Exercise: Prove that all the limit points of T **are in** K.

We return to the example. Consider the sequence $\{e_i\} \subseteq S$ where e_i is the i-th "coordinate vector" of ℓ^2: $e_1 = (1, 0, 0, 0, \ldots)$ and if $i > 1$, $e_i = (0, 0, 0, \ldots, 0, 1, 0, 0, 0, \ldots)$ with $i - 1$ zeroes preceding a 1 in coordinate i, followed by all zeroes.

If S were compact, $\{e_i\}$ would have a convergent subsequence $\{e_{i_k}\}$. But if $i \neq j$ $\|e_i - e_j\| = \sqrt{2}$ so no subsequence is Cauchy, and S cannot be compact.

Review Exercise Prove that if T has a limit point p_o then there is a sequence of distinct points in T that converges to p_o.

Review Exercise Prove in more detail the statement "every point $x \in K$ has an open neighborhood U_x that contains no point of T except, possibly, x itself" that appears in the proof of the Bolzano-Weierstrass Theorem.