**The Cauchy Mean Value Theorem, or Generalized mean Value Theorem**

**Theorem:** *Suppose that* $F(x)$ *and* $G(x)$ *are both continuous on* $[a, b]$ *and both differentiable on* $(a, b)$, *where* $a < b$ *and both are finite. In addition, suppose that* $G'(x) \neq 0$ *on* $(a, b)$. *Then there exists* $c$, $a < c < b$, *such that*

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}.$$

If $G(x) = x$, this is the usual Mean Value Theorem. The proof of this is essentially the same as the proof of the ususal Mean Value Theorem. It relies on Rolle's Lemma.

**Rolle's Lemma:** *Suppose that* $H(x)$ *is continuous on* $[a, b]$ *and differentiable on* $(a, b)$. *In addition, suppose that* $H(a) = H(b)$. *Then there exists* $c$, $a < c < b$, *such that* $H'(c) = 0$.

To prove Rolle's Lemma we need to know that every continuous function on $[a, b]$ takes on maximum and minimum values.

If the maximum value of $H(x)$ and its minimum value are the same, then $H(x) = H(a)$ throughout $[a, b]$, $H'(x) = 0$ on $(a, b)$, and thus we can let $c = (a + b)/2$, or choose any $c \in (a, b)$.

Otherwise $H(x)$ has maximum value $M$ at some $c$, $a < c < b$, that is larger than $H(a) = H(b)$, or else $H(x)$ has minimum value $m$ at $c$ that is smaller than $H(a) = H(b)$.

Let's assume $H(c) = m < H(a) = H(b)$, and $H(c) \leq H(x)$ for all $x \in [a, b]$. Then $H'(c) = \lim_{x\to c} \frac{H(x) - H(c)}{x - c}$.

$$\text{Since } a \leq x < c \Rightarrow \frac{H(x) - H(c)}{x - c} = \frac{+}{-} \leq 0 \text{ and } c < x \leq b \Rightarrow \frac{H(x) - H(c)}{x - c} = \frac{+}{+} \geq 0,$$

we must have $H'(c) = \lim_{x\to c} \frac{H(x) - H(c)}{x - c} = 0$.

To use this to prove the Cauchy Mean Value Theorem we need to invent a function $H(x)$ that has the same values at $a$ and $b$, that includes $F(x)$ and $G(x)$ somehow, and that causes the equation in the statement of the Cauchy Mean Value Theorem to be true. We need to know that $G(b) \neq G(a)$ because we want to divide by $G(b) - G(a)$. That's where assuming $G'(x) \neq 0$ on $(a, b)$ comes in. Let's take this for granted right now.

There are several possibilities for $H(x)$. One thing to try might be $(F(x) - F(a))(G(b) - G(x))$. At least it has the same value at both ends, and Rolle's Lemma applies. But it does not work.

Here's our $H(x)$: $H(x) = (F(x) - F(a))(G(b) - G(a)) - (F(b) - F(a))(G(x) - G(a))$. We have $H(a) = 0 = H(b)$, $H$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Hence by Rolle's Lemma there is some $c$ strictly between $a$ and $b$ such that $H'(c) = 0$. Then

$$H'(c) = F'(c)(G(b) - G(a)) - (F(b) - F(a))G'(c) = 0 \text{ so } \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}, \text{ for } G(b) \neq G(a).$$

To see why assuming $G'(x) \neq 0$ in $(a, b)$ implies that $G(b) \neq G(a)$, we apply the result we just proved, first to the pair $G(x)$ and $x$. Now, with $x$ in the rôle of $G(x)$, we see at once that our denominator can't be zero, so that $\frac{G(b)-G(a)}{b-a} = \frac{G'(c')}{1} \neq 0$ (i.e., we apply the regular MVT).

**An application: Taylor's Theorem with Remainder**

**Taylor's Theorem with Remainder:** *Suppose that* $f(x)$ *is continuous on* $(a, b)$ *and that for* $1 \leq n \leq N + 1$, $f^{(n)}(x)$ *exists on* $(a, b)$. *Then for each* $x_o \in (a, b)$ *and each* $x \in (a, b)$, *there exists* $c_N \in (a, b)$, *strictly between* $x$ *and* $x_o$, *such that*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n + \frac{f^{(N+1)}(c_{N+1})}{(N + 1)!}(x - x_o)^{N+1}.$$

The quantity $\frac{f^{(N+1)}(c_{N+1})}{(N+1)!}(x-x_o)^{N+1}$ is the "Remainder." The Theorem and its proof don't tell us what $c_{N+1}$ is, because we'll use the Cauchy MVT (several times!) and that Theorem doesn't tell us what it's $c$ is. In the CMVT we will use $t$ as the variable, because our application uses $x$ as a quantified variable – i.e., $x$ is "pre-empted."

To prove this Theorem we will take $F(t) = f(t) - \sum_{n=0}^{N} \frac{f^{(n)}(x_o)}{n!}(t-x_o)^n$ and we will set $G(t) = (t-x_o)^{N+1}$. Let us notice that $F(x_o) = 0 = G(x_o)$. We first apply the CMVT to $F(t)$ and $G(t)$ on the interval $[x_o, x]$ if $x_o < x$, or to the interval $[x, x_o]$ if $x_o > x$. If $x_o = x$, there is nothing to prove. We may assume that $x_o < x$, for the same argument would be used in the other case. On the interval $(x_o, x)$ the function $G'(t) \neq 0$, and indeed it is true that $G^{(k)}(t) = \frac{(N+1)!}{(N+1-k)!}(t-x_o)^{N+1-k} \neq 0$ on $(x_o, x)$ as along as $k \leq N+1$ (we'll have to use that). The CMVT continuity and differentiability conditions are true on our intervals $[x_o, x]$ and $(x_o, x)$ since $[x_o, x] \subseteq (a, b)$. Thus there exists $c_1 \in (x_o, x)$ such that

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(x_o)}{G(x) - G(x_o)} = \frac{F'(c_1)}{G'(c_1)}.$$

Let's look at what we have. First,

$$F'(t) = f'(t) - \sum_{n=1}^{N} \frac{f^{(n)}(x_o)}{n!} n(t-x_o)^{n-1} = f'(t) - \sum_{n=0}^{N-1} \frac{f^{(n+1)}(x_o)}{n!}(t-x_o)^n = f'(t) - \sum_{n=0}^{N-1} \frac{(f')^{(n)}(x_o)}{n!}(t-x_o)^n,$$

so $F'(t)$ is the "$F$" for $f'(t)$, with $N$ reduced by one. In particular, $F'(x_o) = 0$ (we set $t = x_o$). We already calculated the derivatives of $G(t)$, and $G'(x_o) = 0$ too. Now we can write

$$\frac{F(x)}{G(x)} = \frac{F(x) - F(x_o)}{G(x) - G(x_o)} = \frac{F'(c_1)}{G'(c_1)} = \frac{F'(c_1) - F'(x_o)}{G'(c_1) - G'(x_o)}.$$

You can guess now that we're going to apply CMVT again, but now to the interval $[x_o, c_1]$, getting $c_2 \in (x_o, c_1)$ such that

$$\frac{F(x)}{G(x)} = \frac{F'(c_1) - F'(x_o)}{G'(c_1) - G'(x_o)} = \frac{F''(c_2)}{G''(c_2)} = \frac{F''(c_2) - F''(x_o)}{G''(c_2) - G''(x_o)},$$

and so on. At this point tho, it's a good idea to let "notation" come to the rescue.

We define the *Taylor polynomial of* $f(x)$, *of degree* $N$, *at the point* $x_o$, and denote it $T_N[f](x)$, by the formula

$$T_N[f](x) := \sum_{n=0}^{N} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n.$$

**Exercise:** Verify that $\frac{d}{dx}\Big[T_N[f](x)\Big] = T_{N-1}[f'](x)$. Thus you show that $\left(\frac{d}{dx}\right)^k \Big[T_N[f](x)\Big] = T_{N-k}[f^{(k)}](x)$, as long as $k \leq N$. In particular, check that $\left(\frac{d}{dx}\right)^N \Big[T_N[f](x)\Big] = f^{(N)}(x_o)$.

With this notation we can see easily that

$$F^{(k)}(t) = \left(\frac{d}{dt}\right)^k F(t) = f^{(k)}(t) - T_{N-k}[f^{(k)}](t) \text{ and that } F^{(k)}(x_o) = 0.$$

Hence

$$\frac{F(x)}{G(x)} = \frac{F'(c_1) - F'(x_o)}{G'(c_1) - G'(x_o)} = \cdots = \frac{F^{(k)}(c_k) - F^{(k)}(x_o)}{G^{(k)}(c_k) - G^{(k)}(x_o)} = \cdots = \frac{F^{(N)}(c_N) - F^{(N)}(x_o)}{G^{(N)}(c_N) - G^{(N)}(x_o)} = \frac{F^{(N+1)}(c_{N+1})}{G^{(N+1)}(c_{N+1})}.$$

But since $\left(\frac{d}{dx}\right)^{N+1} \Big[T_N[f](x)\Big] = \left(\frac{d}{dx}\right) f^{(N)}(x_o) = 0$, and $G^{(N+1)}(t) = (N+1)!$,

$$\frac{F^{(N+1)}(c_{N+1})}{G^{(N+1)}(c_{N+1})} = \frac{f^{(N+1)}(c_{N+1})}{(N+1)!}, \text{ so } F(x) = \frac{f^{(N+1)}(c_{N+1})}{(N+1)!}G(x) = \frac{f^{(N+1)}(c_{N+1})}{(N+1)!}(x-x_o)^{N+1}. \text{ Done!}$$