

**Theorem** If  $K$  is a subset of a metric space  $X$ , then  $K$  is compact set if and only if every infinite subset of  $K$  has a limit point in  $K$ .

*Proof:* In the first part of the proof, we will not need  $X$  to be a metric space. Thus suppose that  $K$  is a compact set in a topological space. Let  $E$  be a subset of  $K$  that has no limit points in  $K$ . This means that, for every point  $x \in K$  there is an open set  $\mathcal{N}_x$  containing  $x$  such that  $\mathcal{N}_x \cap E \subseteq \{x\}$ . The sets  $\mathcal{N}_x$  comprise an open covering of  $K$ , so there are finitely many points  $x_1, \dots, x_N$  such that  $\{\mathcal{N}_{x_1}, \dots, \mathcal{N}_{x_N}\}$  is an open cover of  $K$ . But then

$$E = E \cap K \subseteq E \cap \bigcup_{j=1}^N \mathcal{N}_{x_j} \subseteq \{x_1, \dots, x_N\},$$

so  $E$  is finite if  $E$  has no limit points in  $K$ . This completes the first part of the proof.

To prove the second part we return to the assumption that  $X$  is a metric space, and we will denote the metric by  $d(x, y)$ . We are now given that  $K$  is a subset of  $X$  with the property that every infinite subset of  $K$  has a limit point in  $K$ . To show that  $K$  is compact we suppose that  $\mathcal{U}$  is an open cover of  $K$  and we define a function  $\delta(x)$ , for  $x \in K$ , that tells "how well" a point  $x \in K$  is covered by  $\mathcal{U}$ :

$$\text{for } x \in K, \quad \delta(x) := \sup\{0 < r < 1 : B_r(x) \subseteq U \text{ for some } U \in \mathcal{U}\}.$$

The function  $\delta(x)$  is well-defined because the set  $\{0 < r < 1 : B_r(x) \subseteq U \text{ for some } U \in \mathcal{U}\}$  is bounded by construction, and is non-empty since  $\mathcal{U}$  is an open cover of  $K$ . Moreover,  $\delta(x) > 0$  for all  $x \in K$ .

There are two cases to consider:  $\delta(x)$  is, or is not, bounded below by some small positive number.

If, for all  $x$  in  $K$ ,  $\delta(x) \geq 2\epsilon > 0$ , we can prove that the open cover  $\{B_\epsilon(x) : x \in K\}$  has a finite subcover. This will prove that the original open cover has a finite subcover, because each ball  $B_\epsilon(x)$  is contained in a set  $U \in \mathcal{U}$ .

We proceed by assuming that  $\delta(x) \geq \epsilon > 0$  for all  $x$  in  $K$ , but that  $\{B_\epsilon(x) : x \in K\}$  has no finite subcover. Let us choose some point  $x_1 \in K$  to start. Then  $K \setminus B_\epsilon(x_1) \neq \emptyset$  because otherwise  $K$  could be covered by finitely many sets from  $\{B_\epsilon(x) : x \in K\}$ . We choose  $x_2 \in K \setminus B_\epsilon(x_1)$ . Then  $d(x_1, x_2) \geq \epsilon$  and as before  $K \setminus B_\epsilon(x_1) \cup B_\epsilon(x_2) \neq \emptyset$ . This allows us to construct, by induction, a sequence  $\{x_n\}$  of points in  $K$  such that  $d(x_k, x_\ell) \geq \epsilon$  whenever  $k \neq \ell$ . Since the points in the sequence  $\{x_n\}$  are distinct, the points in  $\{x_n\}$  comprise an infinite subset of  $K$  and so there exists a limit point  $x_\omega \in K$  for  $\{x_n\}$ . Since  $B_{\epsilon/3}(x_\omega)$  contains infinitely many points of  $\{x_n\}$ , there exists  $k \neq \ell$  such that  $\{x_k, x_\ell\} \subseteq B_{\epsilon/3}(x_\omega)$ . But then by the triangle inequality we have the contradiction

$$\epsilon \leq d(x_k, x_\ell) \leq d(x_k, x_\omega) + d(x_\omega, x_\ell) < \epsilon/3 + \epsilon/3 = 2\epsilon/3.$$

Thus our set  $\{x_n\}$  could not be infinite, so for some  $N$ ,  $K \setminus (B_\epsilon(x_1) \cup \dots \cup B_\epsilon(x_N)) = \emptyset$ , which is another way of saying  $K \subseteq B_\epsilon(x_1) \cup \dots \cup B_\epsilon(x_N)$ .

Now let us turn to the other case, in which there is no  $\epsilon > 0$  such that  $\delta(x) \geq \epsilon$  for all  $x \in K$ . We will actually show that this case cannot happen. But suppose it does. Then we can find a point  $x_0$  such that  $0 < \delta(x_0) < 1$ . We can then find  $x_1$  such that  $0 < \delta(x_1) < \frac{1}{2}\delta(x_0) < \frac{1}{2}$ . By induction, we can find a sequence  $\{x_n\}$  of points in  $K$  such that  $0 < \delta(x_k) < \frac{1}{2}\delta(x_{k-1}) < \frac{1}{2^k}$ . Since the points in the sequence  $\{x_n\}$  are distinct, the points in  $\{x_n\}$  comprise an infinite subset of  $K$  and so there exists a limit point  $x_\omega \in K$  for  $\{x_n\}$ . Then the ball  $B_{\delta(x_\omega)/8}(x_\omega)$  contains infinitely many points of  $\{x_n\}$  so it contains some  $x_k$  with  $0 < \delta(x_k) < 2^{-k} < \delta(x_\omega)/8$ .

Then  $2\delta(x_k) < \delta(x_\omega)/4$ , so

$$B_{2\delta(x_k)}(x_k) \subseteq B_{\delta(x_\omega)/2}(x_\omega) \subseteq U \text{ for some } U \in \mathcal{U} \text{ (if } y \in B_r(x) \text{ then } B_r(y) \subseteq B_{2r}(x)\text{)}.$$

This contradicts the definition of  $\delta(x_k)$ . Thus the case we have been considering is impossible. This completes the proof of the Theorem.

**Remark!** We have also shown that every open covering  $\mathcal{U}$  of a compact set  $K$  in a metric space has a *Lebesgue number*, namely there exists an  $\epsilon > 0$  such that for every  $x \in K$ ,  $B_\epsilon(x) \subseteq U$  for some  $U \in \mathcal{U}$ .