If a sequence $\{x_n\}$ converges, the terms in the sequence get closer and closer to the limit of the sequence. Thus they have to "eventually" stay close together. This is the idea behind the Cauchy Convergence Criterion:

A sequence $\{x_n\}$ *converges if and only if for every* $\epsilon > 0$ *there exists an integer* N *such that for every pair of integers* m, n, *if* $m \geq N$ *and* $n \geq N$ *then* $|x_m - x_n| < \epsilon$.

Sequences that satisfy the condition in the Cauchy Convergence Criterion are called *Cauchy sequences.* We can restate the Cauchy Convergence Criterion as "A sequence is convergent if and only if it is Cauchy."

In this course we are not using the Completeness Axiom as such, so we need a different "power source" to do that job. We use the Monotone Sequence Theorem, which states that a bounded monotone sequence has a limit, but we only need it in the part of the proof that starts with the assumption that a sequence is Cauchy and is supposed to lead to the conclusion that the sequence converges. That's the hard part.

Let's do the easy part first: show that if a sequence converges then it is a Cauchy sequence. What follows is the long version of the easy part. The short version will follow.

Exercise: Before reading further, try to remember how we did this in class – at least the *idea.*

We're given that $\{x_n\}$ converges. That means that there exists a real number L such that $\lim_{n\to\infty} x_n = L$. But what does "$\lim_{n\to\infty} x_n = L$" *really* mean? It means that for all $\epsilon > 0$ there exists N so that $n \geq N \Rightarrow |x_n - L| < \epsilon$.

What are *we* supposed to do? We're supposed to show that for every $\epsilon > 0$ there exists an integer N such that for every pair of integers m, n, if $m \geq N$ and $n \geq N$ then $|x_m - x_n| < \epsilon$.

Let's do it: we begin by dividing our *given* ϵ by two and then we use $\epsilon/2$ as the ϵ in "$\lim_{n\to\infty} x_n = L$."

Thus we know there exists N such that $n \geq N \Rightarrow |x_n - L| < \epsilon/2$.

We will now choose, as the N to propose in the Cauchy-sequence definition, this same N, the one that corresponds to $\epsilon/2$ in the definition of convergence to L of $\{x_n\}$.

Now we see what happens if we assume that $m \geq N$ and $n \geq N$:

$$|x_n - x_m| \leq |x_n - L| + |L - x_m| = |x_n - L| + |x_m - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

That's it; we're done, because we have shown that our sequence is a Cauchy sequence.

Short version of easy part

Since $\{x_n\}$ converges we know there exists a real number L such that $\lim_{n\to\infty} x_n = L$. We let $\epsilon/2$ play the role of ϵ in convergence to L, and find N such that $n \geq N \Rightarrow |x_n - L| < \epsilon/2$. If, now, we have $m \geq N$ and $n \geq N$ then

$$|x_n - x_m| \leq |x_n - L| + |L - x_m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The hard part

We will need to use a Lemma that is a useful Theorem usually called the Nested Intervals Theorem.

Lemma: *Suppose that* $\{I_n\}$ *is a sequence of bounded closed non-empty intervals that is nested, namely* $I_{n+1} \subseteq I_n$ *for all* n. *Also suppose that the lengths of the intervals* I_n *converge to zero. Then, there exists a unique real number* c *that is in every one of the intervals* I_n.

Example If we let $I_n = [0, 1/n]$ then these intervals are nested, and their lengths tend to zero. The numer 0 is in all the intervals, and it's the only one that is.

Let's use the Lemma first and prove it later. If you wish, you can read the proof first – look for "Proof of the Lemma" toward the end of these notes.

Now we'll assume that $\{x_n\}$ is a Cauchy sequence; we have to find a limit for it to converege to. That's what we'll use the Lemma for (the Lemma asserts the existence of a number, and we'll use that for the limit).

We will use infinitely many epsilons: for each positive integer k we set $\epsilon_k := 1/k$. For each of these epsilons we can apply the definition of Cauchy sequence: there exists M_k such that for every pair of integers m, n, if $m \geq M_k$ and $n \geq M_k$ then $|x_m - x_n| < 1/k$.

We will let $m = M_k$. Since we want to use the Lemma we need to convert the inequality $|x_{M_k} - x_n| < 1/k$ into a statement about an interval. The inequality says that $x_n \in [x_{M_k} - 1/k,\, x_{M_k} + 1/k]$. This is true for every $n \geq M_k$.

It would really be nice if we could use the Lemma right now, but we have no way of knowing that the intervals $[x_{M_k} - 1/k,\, x_{M_k} + 1/k]$ are nested! We *do* know their lengths tend to zero, but we need both things to be true!

We will build nested intervals out of the ones we have in two steps.

The first step: replace the integers M_k by new, bigger integers N_k chosen so that $N_{k+1} > N_k$. The reason for doing this is that (before) for every $n \geq M_k$ we knew $x_n \in [x_{M_k} - 1/k,\, x_{M_k} + 1/k]$. But we also need to know that $n \geq M_{k+1}$ means that $n \geq M_k$ as well! We'll see why soon.

We now define

$$N_k := \max_{1 \leq j \leq k} M_j + k.$$

To see that $N_{k+1} > N_k$ we observe in passing that $\max\limits_{1 \leq j \leq k} M_j = N_k - k$. Then we look at N_{k+1} this way:

$$N_{k+1} = \max_{1 \leq j \leq k+1} M_j + k + 1 = \max\left\{ \max_{1 \leq j \leq k} M_j,\, M_{k+1} \right\} + k + 1 = \max\{N_k - k,\, M_{k+1}\} + k + 1.$$

Now the maximum of two numbers is at least as large as each of them, so we know that $\max\{N_k - k,\, M_{k+1}\} \geq N_k - k$. Thus when we put this inequality into our last expression for N_{k+1} we get

$$N_{k+1} = \max\{N_k - k,\, M_{k+1}\} + k + 1 \geq (N_k - k) + k + 1 = N_k + 1, \quad \text{hence} \quad N_{k+1} > N_k.$$

Just as before, with the M_k's, we know that for **every** $n \geq N_k$, $x_n \in [x_{N_k} - 1/k,\, x_{N_k} + 1/k]$. All that we're doing differently here is picking $m = N_k$ now instead of $m = M_k$. The big difference now is that we know that, for example, if $n \geq N_{k+1000}$, it's still true that $x_n \in [x_{N_k} - 1/k,\, x_{N_k} + 1/k]$.

The second step: We still don't know that the intervals we just made are nested. But we can make use of this fact: *the intersection of two closed intervals is a closed interval.* So first let's define $J_k := [x_{N_k} - 1/k,\, x_{N_k} + 1/k]$. Now we're ready to start: Let's let $I_1 = J_1$, $I_2 = J_1 \cap J_2$, $I_3 = J_1 \cap J_2 \cap J_3$, and in general, let

$$I_n := \bigcap_{k=1}^{n} J_k = J_1 \cap J_2 \cap \cdots \cap J_n.$$

Thus I_n is, by induction, a closed interval. Is I_n non-empty? Yes, because every x_n with $n \geq N_n$ is in I_n. That needs to be proved by induction, because we have to know that every x_m that is in $[x_{N_n} - 1/n,\, x_{N_n} + 1/n]$ (because $m \geq N_n$) is also in every one of the intervals J_k with $k \leq n$. But that's true because if $x_m \geq N_n$ then $x_m \geq N_k$ as well, so $x_m \in J_k$. The I_n are nested too, because (a fact about sets) $A \cap B \subseteq A$ is always true. This plus an induction shows that the I_n are nested.

We have to show also that the lengths of the $I_n \to 0$. Well, we know that J_n has length $2/n \to 0$, and we know that $I_n \subseteq J_n$, so the length of $I_n \to 0$ by the Squeeze Principle.

Thus by the Lemma there is a unique number c that is in all the intervals I_n.

Let us set $L := c$ and show that $x_n \to L$. Let $\epsilon > 0$ be given. We choose m so large that $2/m < \epsilon$. Then we know that every x_n with $n \geq N_m$ is in I_m and we know that $L = c \in I_m \subseteq I_{N_m}$. Accordingly, $|x_n - L|$ can't be any bigger than the length of I_m, which is at most $2/m < \epsilon$. We have done it! We have shown that for our given $\epsilon > 0$, if $n \geq N_m$, where m is an integer bigger than $2/\epsilon$, then $|x_n - L| < \epsilon$.

Proof of the Lemma

We are given that the intervals I_n are nonempty and closed and bounded. Therefore each interval I_n can be written $I_n = [a_n, b_n]$, where $a_n \le b_n$. If we were to draw I_n and then draw I_{n+1} in the same picture, on the same number line, where would a_{n+1} be, relative to a_n? It would be to the right of a_n. Otherwise, we'd have $a_{n+1} < a_n$ and then the number a_{n+1} would not belong to $[a_n, b_n]$. This establishes that $a_n \le a_{n+1}$ for all n. In a similar manner we show that $b_{n+1} \le b_n$ for all n. This gives us two monotone sequences, $\{a_n\}$ and $\{b_n\}$, with the first increasing and the second decreasing. Are they bounded?

Let's first check that $\{a_n\}$ is bounded below. It's bounded below by a_1. Similarly, $\{b_n\}$ is bounded above by b_1.

Now to check that $\{a_n\}$ is bounded above we use the connection between the a's and the b's: $a_n \le b_n \le b_1$, so $\{a_n\}$ is bounded above by b_1.

We also have $b_n \ge a_n \ge a_1$, so $\{b_n\}$ is bounded below by a_1. Therefore, by the Monotone Sequence Theorem we see that there exist real numbers a and b such that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. But we are given that $b_n - a_n \to 0$, so $b - a = \lim_{n\to\infty} b_n - a_n = 0$. We set $c := a = b$.

Now, to show that $c \in I_n$ for all n, we notice that $a_n \le a = c$ and $b_n \ge b = c$, so we combine the inequalities and get: $a_n \le c \le b_n$ for all n. But as we have seen, $a_n \le c \le b_n$ means the same as $c \in [a_n, b_n] = I_n$, and this is true for all n.

Finally we need to show that c is unique. To show this we assume that $c' \in I_n$ for all n, and we have to show that $c' = c$.

Since both of them belong to I_n, we know that $|c - c'| \le b_n - a_n$ for all n (why?). Since $b_n - a_n \to 0$, we have to have $c' = c$.

The proof is done.

Exercises: Every time you encounter a question herein, try to answer it yourself before continuing.

Examples

Two very important examples illustrate the use of the Cauchy sequence idea.

The first example is called the Harmonic Series. We will show that the sequence of partial sums is not a Cauchy sequence, so the sequence of partial sums does not converge and thus the series diverges. Since the sequence of partial sums is increasing this shows that the sequence of partial sums is not bounded above, and this in turn means that the series diverges to plus infinity.

The second example is made from the first by making the signs in the harmonic series alternate. The new series converges. It actually converges to $\ln 2$, but we won't use that information – we'll just show that the sequence of partial sums is a Cauchy sequence.

The Harmonic Series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges to $+\infty$.

Its Alternating Series, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, which converges.

To show that the Harmonic Series diverges, using the Cauchy Criterion, we need the denial of the statement that the sequence of partial sums is a Cauchy sequence. What is the denial? It is: there exists $\epsilon > 0$ such that for all integers N there exist integers $m \ge N$ and $n \ge N$ such that $|h_m - h_n| \ge \epsilon$, where $h_k := \sum_{j=1}^{k} 1/j$, for all positive integers k.

We know there is a positive integer K so large that $2^K > N$. We'll then set $m = 2^K + 1$ and set $n = 2^{K+1}$. Then $|h_n - h_m| = h_n - h_m = \sum_{j=m+1}^{n} 1/j$. Let's notice that $1/j > 1/n$ if $m + 1 \le j < n$. Therefore since there are 2^K terms in the sum (why?), $|h_n - h_m| = \sum_{j=m+1}^{n} 1/j > 2^K/n = 1/2$ (why?). So: we take our $\epsilon = 1/2$ and we have shown that the sequence $\{h_n\}$ is indeed *not* a Cauchy sequence.

The second example is easily handled using the Alternating Series Theorem. However, here we don't have that Theorem yet. What we do have is the Theorem about telescoping series. This allows us to say, for example, that

$$\sum_{n=N}^{\infty} \frac{1}{n(n+1)} = \left(\frac{1}{N} - \frac{1}{N+1}\right) + \cdots + \left(\frac{1}{N+m} - \frac{1}{N+m+1}\right) + \cdots = \frac{1}{N}$$

because the "in between" terms all cancel. We'll use that fact in this proof that the series converges. I suggest you use the Alternating Series Test instead of this proof. This proof is here just to show that we can accomplish a lot with minimal tools.

To show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges we'll examone the partial sums. We let $s_n := \displaystyle\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j}$. There are some cases to consider. If n is even, the terms can be paired in a $+-$ pattern. And n is even means that $n = 2k$ for some integer k. Thus

$$s_n = s_{2k} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) = \sum_{j=1}^{k} \left(\frac{1}{2j-1} - \frac{1}{2j}\right) = \sum_{j=1}^{k} \frac{1}{(2j-1)2j},$$

where $k = n/2$.

We notice that all our paired terms have the form "one-over-odd minus one-over-next even." If we insert the "missing" terms, "one-over-even one-over-next odd," [inserted terms in square brackets] we get

$$s_{2k} < \left(1 - \frac{1}{2}\right) + \left[\frac{1}{2} - \frac{1}{3}\right] + \left(\frac{1}{3} - \frac{1}{4}\right) + \left[\frac{1}{4} - \frac{1}{5}\right] + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left[\frac{1}{2k-2} - \frac{1}{2k-1}\right] + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) < 1.$$

Therfore the sequence $\{s_{2k}\}$ converges, because it is an increasing sequence that is bounded above. Hence $\{s_{2k}\}$ is a Cauchy sequence. **Exercise** Calculate the first four or five terms of the "whole" sequence $\{s_n\}$ so you can see that it's not an monotone sequence.

Note that $\{s_{2k}\}$ is a sequence with variable k. It's as though we *defined* $t_k := s_{2k}$. The sequence $\{s_{2k}\}$ is an example of what is called a *subsequence.*

On the other hand, if n is odd we can write $n = 2k-1$ for some integer k. In *this* case, $s_n = s_{2k-1} = s_{2k} + \frac{1}{2k}$. We could also write $s_n = s_{2k-1} = s_{2k-2} + \frac{1}{2k-1}$. Thus $\{s_{2k-1}\}$ is the sum of an increasing sequence and a decreasing sequence, no matter which way we look at it...

Now we want to show that $\{s_n\}$ is a Cauchy sequence.

We need to look at $s_m - s_n$, where m and n are both large, and there is no connection between the two at all. We have to show that for every given $\epsilon > 0$ there exists N so large that if $m \geq N$ and $n \geq N$ then $|s_m - s_n| < \epsilon$. Let's assume (by renaming if we need to) that $m < n$. Then there are four cases to consider:

$$\begin{aligned} s_n - s_m &= \sum_{j=m+1}^{m} \frac{(-1)^{n+1}}{n} \\ &= \begin{cases} s_{2k} - s_{2\ell} & \text{if } n = 2k \text{ and } m = 2\ell; \\ s_{2k} - s_{2\ell-1} & \text{if } n = 2k \text{ and } m = 2\ell-1; \\ s_{2k-1} - s_{2\ell} & \text{if } n = 2k-1 \text{ and } m = 2\ell; \\ s_{2k-1} - s_{2\ell-1} & \text{if } n = 2k-1 \text{ and } m = 2\ell-1. \end{cases} \end{aligned}$$

Since $\{s_{2k}\}$ is a Cauchy sequence, we know that there exists K so large that if $k \geq K$ and $\ell \geq K$ then $|s_{2k} - s_{2\ell}| < \epsilon/2$. Note that we changed the original ϵ to $\epsilon/2$ in the Cauchy-sequence definition for $\{s_{2k}\}$. We'll need that when we consider the case when at least one of m and n is odd.

Next, $|s_{2k} - s_{2\ell-1}| = |s_{2k} - s_{2\ell} - \frac{1}{2\ell}| \leq |s_{2k} - s_{2\ell}| + \frac{1}{2\ell}$. What happens now if $k \geq K$ and $\ell \geq K$? Well, we still have $|s_{2k} - s_{2\ell}| < \epsilon/2$, and we know that $2\ell \geq 2K$, so we wind up with $|s_{2k} - s_{2\ell}| < \epsilon/2 + 1/2K$. Unfortunately we have no idea how big K is, so we don't know whether or not $1/2K < \epsilon/2$ (or $K > 1/\epsilon$), which is what we'd like. But we still have not chosen our N, so we can choose N to be very big, so big that when we "reparametrize" m and n in terms of k and ℓ, it will cause our K to be replaced by something bigger, say $\widehat{K}$, where also $1/2\widehat{K} < \epsilon/2$. The way we'll do this is to recall that we wrote $n = 2k$ if n was even, and we wrote $n = 2k - 1$ if n was odd, and we treated m similarly. Thus if we choose an N we think of $N = 2\widehat{K}$ if N is even and $N = 2\widehat{K} - 1$ if N is odd. So now we can pick $\widehat{K}$ and figure out what N has to be. We'll choose an integer $\widehat{K} > \max\{K, 1/\epsilon\}$. Then we can just choose $N := 2\widehat{K}$.

Have we gotten too far ahead? We have only dealt with two of our four cases, after all. Maybe we'll have to go back and readjust : (. . .

Before we try to do that, let's just see whether our choice works after all!

If $n = 2k - 1$ and $m = 2\ell$ then $s_{2k-1} - s_{2\ell} = s_{2k} + \frac{1}{2k+1} - s_{2\ell} = s_{2k} - s_{2\ell} + \frac{1}{2k+1}$, so

$$|s_n - s_m| = |s_{2k-1} - s_{2\ell}| = \left|s_{2k} - s_{2\ell} + \frac{1}{2k+1}\right| \le |s_{2k} - s_{2\ell}| + \frac{1}{2k+1} < \epsilon/2 + 1/2k \le \epsilon/2 + 1/2\widehat{K} < \epsilon.$$

So far so good; one more case to check. Here we'll use again our choice that $m < n$, which will translate into $2\ell - 1 < 2k - 1$, so that $2\ell + 1 < 2k + 1$ and thus $1/(2k+1) < 1/(2\ell+1)$.

If $n = 2k - 1$ and $m = 2\ell - 1$ then $s_{2k-1} - s_{2\ell-1} = s_{2k} + \frac{1}{2k+1} - s_{2\ell} - \frac{1}{2\ell+1} = s_{2k} - s_{2\ell} + \frac{1}{2k+1} - \frac{1}{2\ell+1}$, so now

$$\begin{aligned} |s_n - s_m| &= |s_{2k-1} - s_{2\ell-1}| \\ &= \left|s_{2k} - s_{2\ell} + \frac{1}{2k+1} - \frac{1}{2\ell+1}\right| \\ &\le |s_{2k} - s_{2\ell}| + \left|\frac{1}{2k+1} - \frac{1}{2\ell+1}\right| < |s_{2k} - s_{2\ell}| + \frac{1}{2\ell+1} \quad \text{(why?)!} \\ &< \epsilon/2 + 1/2\ell \le \epsilon/2 + 1/2\widehat{K} < \epsilon. \end{aligned}$$

We used a fact in the next-to-last line: if $0 < a < b$ then $|a - b| < b$.

This completes the proof that $\{s_n\}$ is a Cauchy sequence. Thus by the Cauchy criterion, $\{s_n\}$ converges.

As mentioned before, the Alternating Series Test is much, much easier to use for this sequence than the argument just given. This argument is a "low level" one; it used very elementary tools. That's what we usually have to use – details about what we're given to work with – when the use of the Cauchy criterion is needed. It's a tool we don't want to use but that we sometimes need to use, like the Mean Value Theorem.